Basics of strength of materials

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Strength of Materials

- A discipline that deals with stiffness and strength of parts of machines or structures and with the causes of their failure due to external forces.
- The external forces acting on the machine part cause the rise of the internal forces called **stress σ**, and at the same time cause the deformation of the component called **strain ε**
- In some simple components of machine or structure, strain and strain can be calculated based on simplistic intuitive assumptions about the distribution of internal forces and knowledge of the behavior of structural materials determined by basic material tests.
- In our short course we will deal mainly with the calculation of stress and strain in the long slim rods strained by tension, bending, torsion and their combinations. The rods are components widely used in the construction of machines and structures (shafts, beams, bars, etc.).
- We will also talk about multi-axis stresses that arise in more complex components.

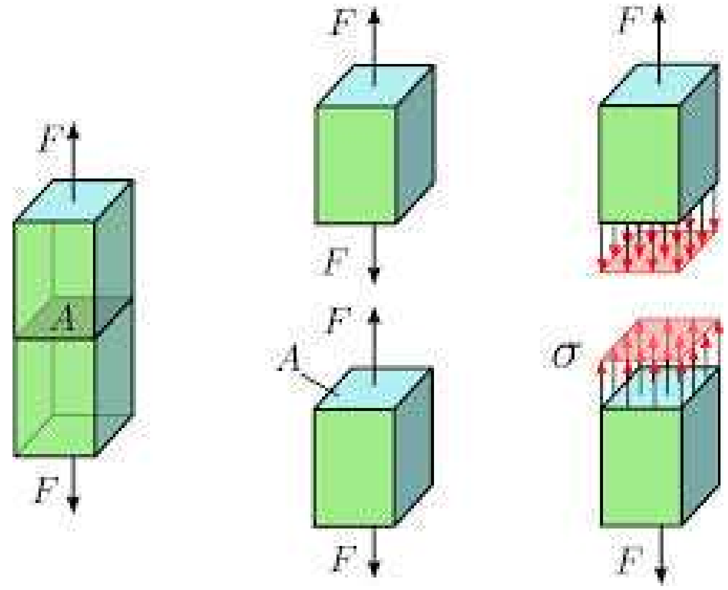
Normal stress and Hooke's law

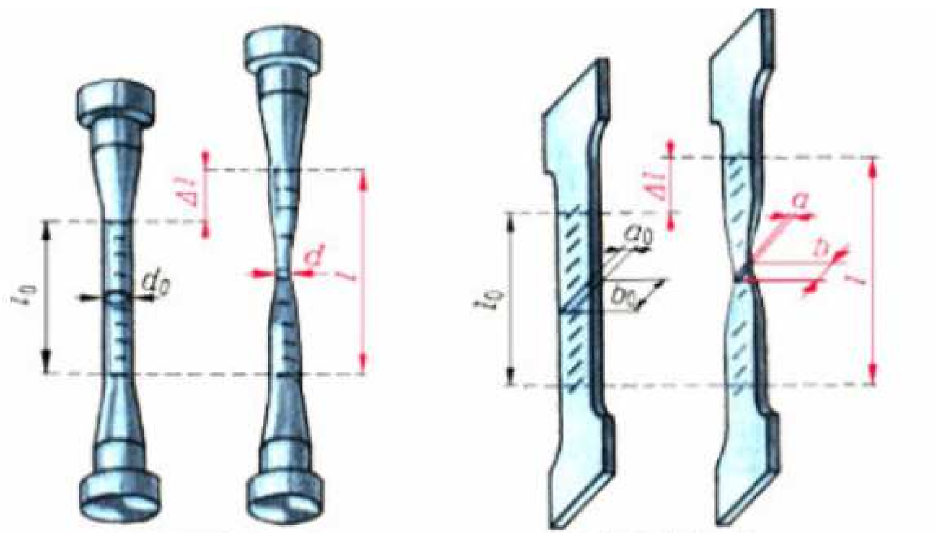
Long slim bars ($l \gg \varnothing d$) strained by pulling forces

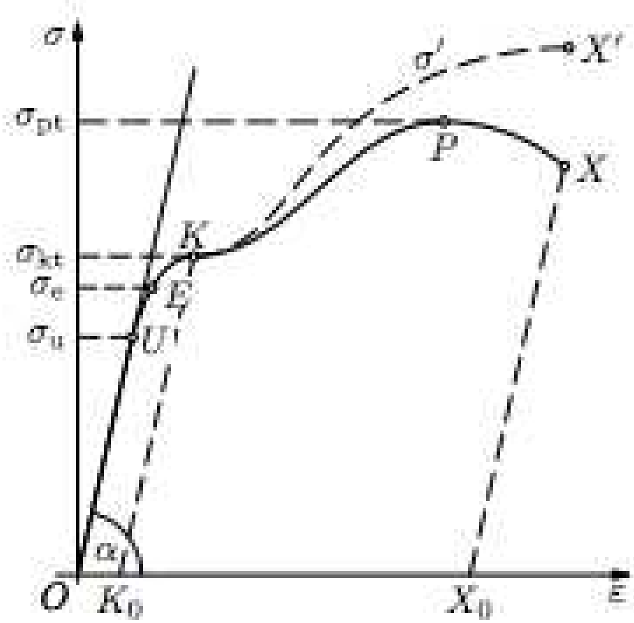
- the assumption that the tension is evenly distributed across the cross-section of the rod (it is constant in the cross-section) - applies only "at a sufficient distance" from the external acting forces and from the shape changes (shoulder, recess, etc.)
- for small deformations the strain is defined as the ratio of the length increment of the rod to its original length
- for the linear part of the tensile diagram, the direct proportionality between tension and strain applies - Hooke's law

$$\sigma = \frac{F}{A}, \quad \varepsilon = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}, \quad \sigma = E\varepsilon,$$

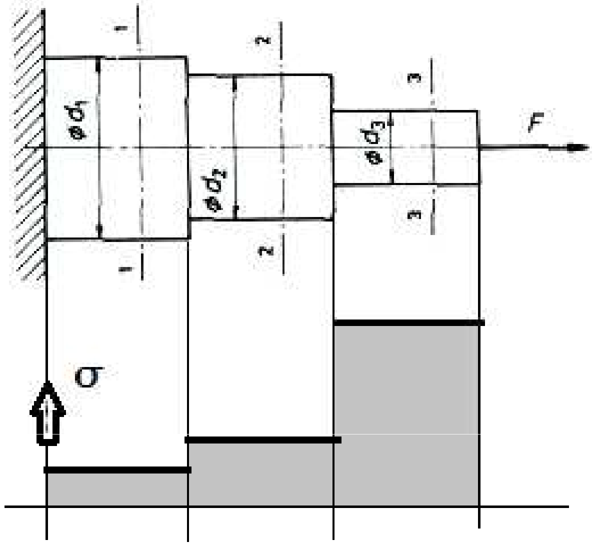
E = Young's modulus of elasticity







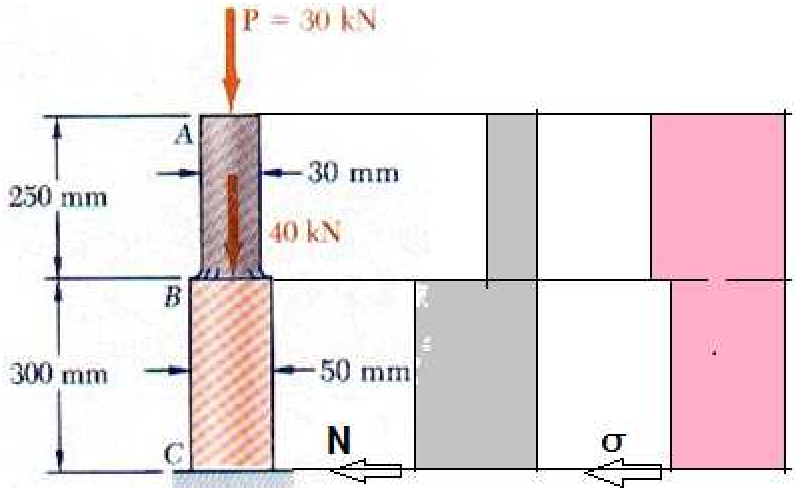
Příklad



Determine stresses in cross-sections 1 -1; 2 -2; 3 – 3 for a bar loaded by the F-axial. Determine the bar extension. ($F = 10^5$ N; $d_1 = 100$ mm; $d_2 = 60$ mm; $d_3 = 30$ mm, $l_1=l_2=l_3=500$mm, E=200000 MPa)

$$\sigma_1/\sigma_2/\sigma_3 = 12,7/35,3/141 MPa$$

$$\Delta l = \frac{\sigma_1}{E} l_1 + \frac{\sigma_2}{E} l_2 + \frac{\sigma_3}{E} l_3 = 0,473 mm$$



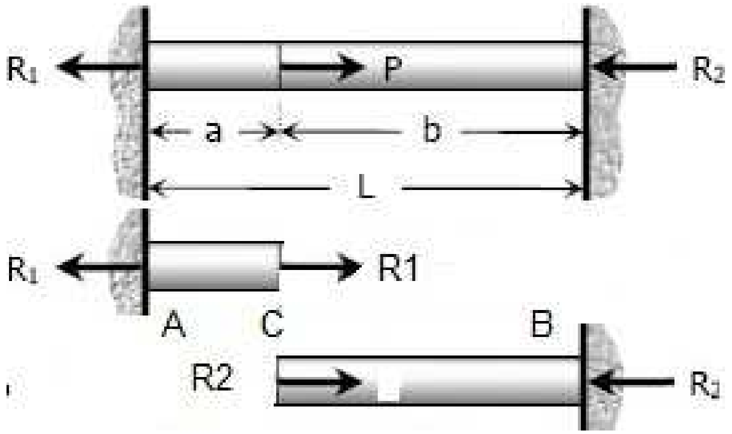
Determine the internal forces and stresses in the pressure-loaded bar, determine its shortening. E=200000Mpa

$$N_{AB}/N_{BC} = -30/-70 kN$$

$$\sigma_{AB}/\sigma_{AC} = -42,4/-35,7 MPa$$

$$\Delta l = \frac{\sigma_{AB}}{E} l_{AB} + \frac{\sigma_{BC}}{E} l_{BC} = -0,107 mm$$

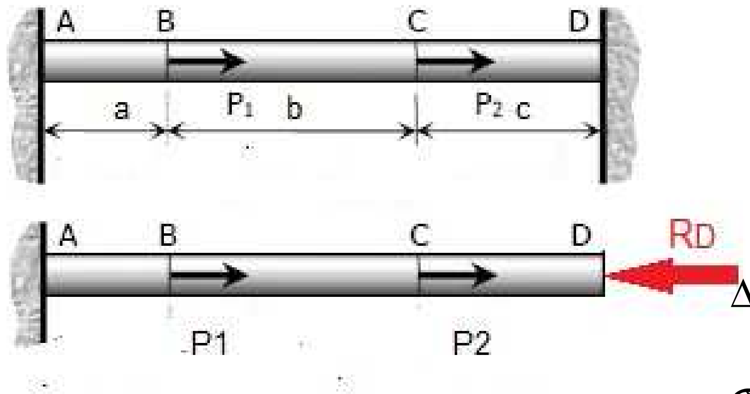
Statically indeterminate cases - tension



The rod is built in the rigid walls at the ends A and B (the length of the rod can not change due the load ΔL = 0), given P = 10kN, Ød = 20mm, a = 200mm, b = 350mm, E = 200000 MPa. **?Stress**

The load force P is decomposed into parts AC and CB in proportion to their stiffness. Equilibr.condition : R1 + R2 = P, elongation of the AC part must be the same as shortening part of CB:

$$\Delta_{AC} = \left|\Delta_{CB}\right| \Rightarrow \frac{R_1 a}{ES} = \frac{R_2 b}{ES} \Rightarrow R_1 = \frac{b}{a} R_2, \sigma_{AC} = 20{,}3MPa, \sigma_{CB} = -11{,}6MPa.$$

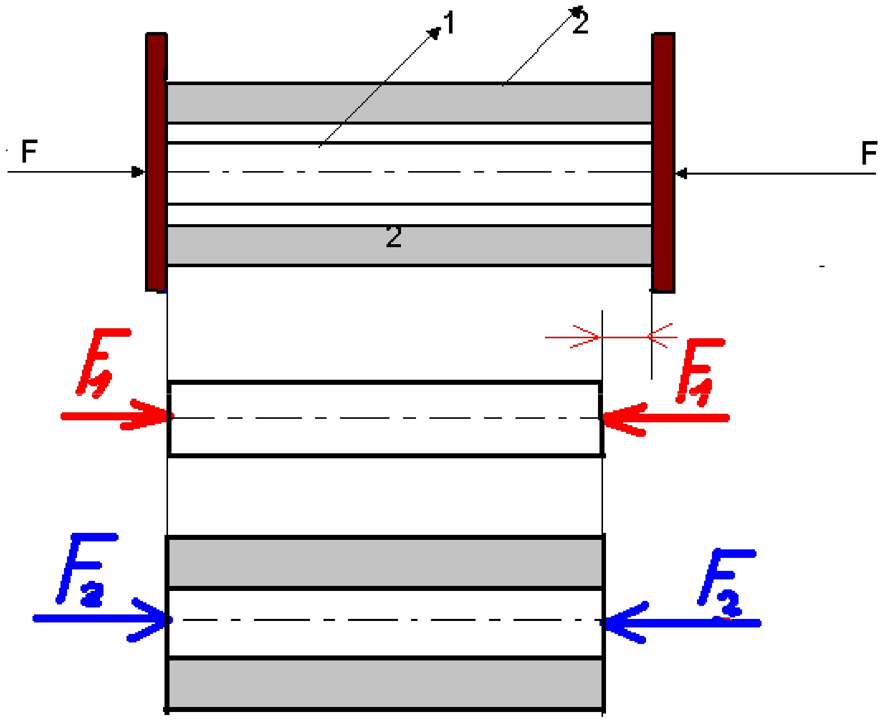


The rod is built in the rigid walls at the ends A and B (the length of the rod can not change due the load ΔL = 0), the extension of the rod due to the forces P1 and P2 must be the same as the reduction of length due to the reaction RD:

$$\Delta L = 0 \Rightarrow \frac{P_2 b}{ES} + \frac{\left(P_1 + P_2\right) a}{ES} - \frac{R_D\left(a+b+c\right)}{ES} = 0 \Rightarrow R_D = \frac{P_1 a + P_2\left(a+b\right)}{\left(a+b+c\right)}$$

$$\sigma_{AB} = \left(P_1 + P_2 - R_D\right)/S,\ \sigma_{BC} = \left(P_2 - R_D\right)/S,\ \sigma_{CD} = -R_D/S.$$

Statically indeterminate cases - tension



The solid rod 1 and the tube 2 are connected by rigid faces and loaded with force F. A portion of the force F1 transmits the rod and the tube transmits a portion of the force F2. Condition of equilibrium: $F_1+F_2=F$

The shortening of both parts must be the same: $\Delta L_1= \Delta L_2$

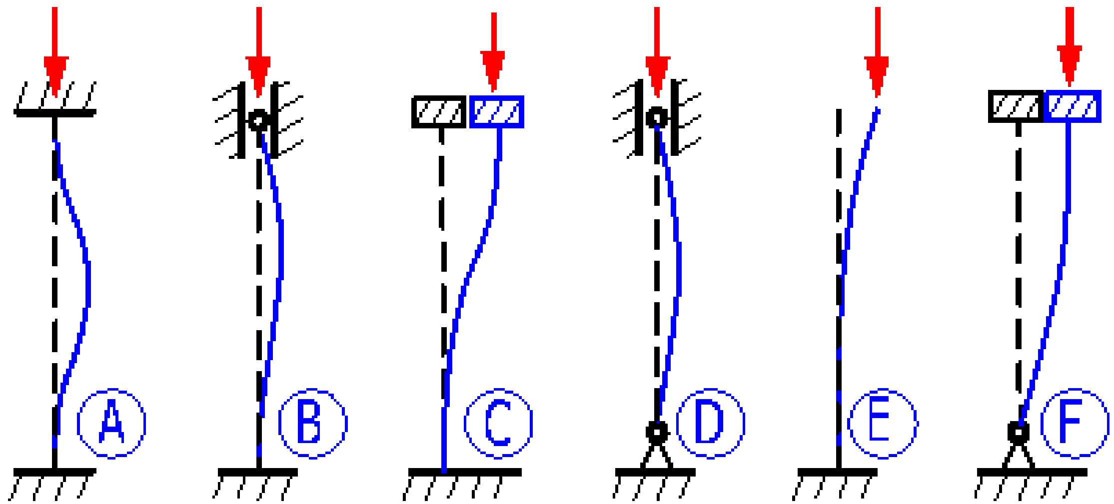
$$F_1L\,/\,(E_1S_1)=F_2L\,/\,(E_2S_2)$$

E=Young modulus
S_1,S_2=cross-sections of rod and tube

Buckling of long slender rods

If the pressure force acts on a long, slim strut, it may deviate due to misalignment or accidental lateral loads. In such cases, the compressive force must not exceed Euler's critical force F_{krit}, which depends on the modulus of elasticity of the material E, on the smallest axial quadratic moment of cross-section J_{min}, on the reduced length of the rod l_{red} and on the type of supports A –B.

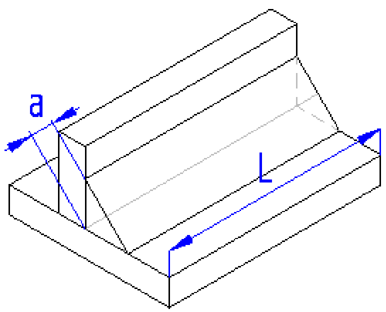
$$F_{krit} = \frac{\pi^2 E J_{\min}}{l_{red}^2}, \; l_{red} = n_k l, \; n_A = 0.5, n_B = 0.7, n_C = n_D = 1, n_E = n_F = 2$$

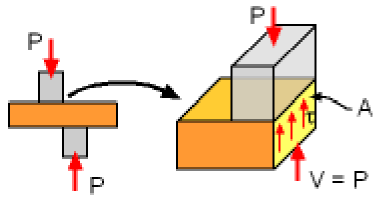


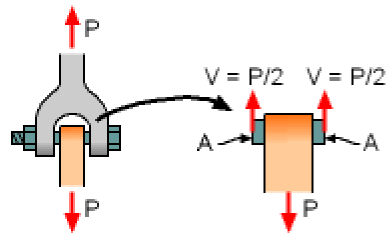
Covenanted calculation of stresses in "cut" and "puncture"

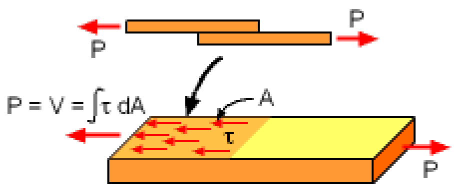
- The simple shear theory is used in technical practice to calculate stresses in rivets, bolts, nails, welds, etc.
- It is assumed that the shear stress is evenly distributed across the cross-section
- The stamp test is based on the assumption that the pressure is evenly distributed over the projection of the pin contact surface
- For edge weldings of thickness **t** we check the shear stress according to the relationship

$$\tau = \frac{T}{a \cdot L}, \; a = \frac{\sqrt{2}}{2} t,$$

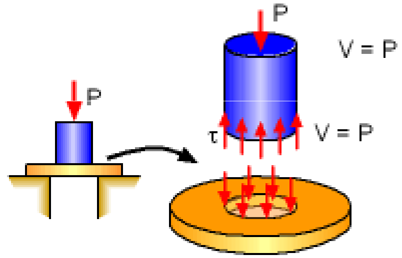


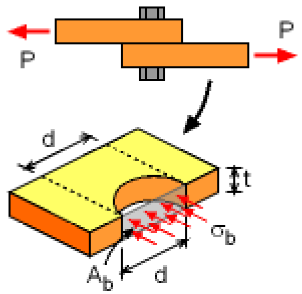




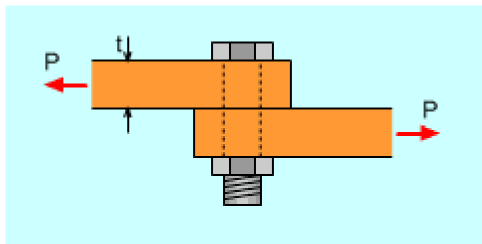


$$\tau = \frac{V}{A}$$





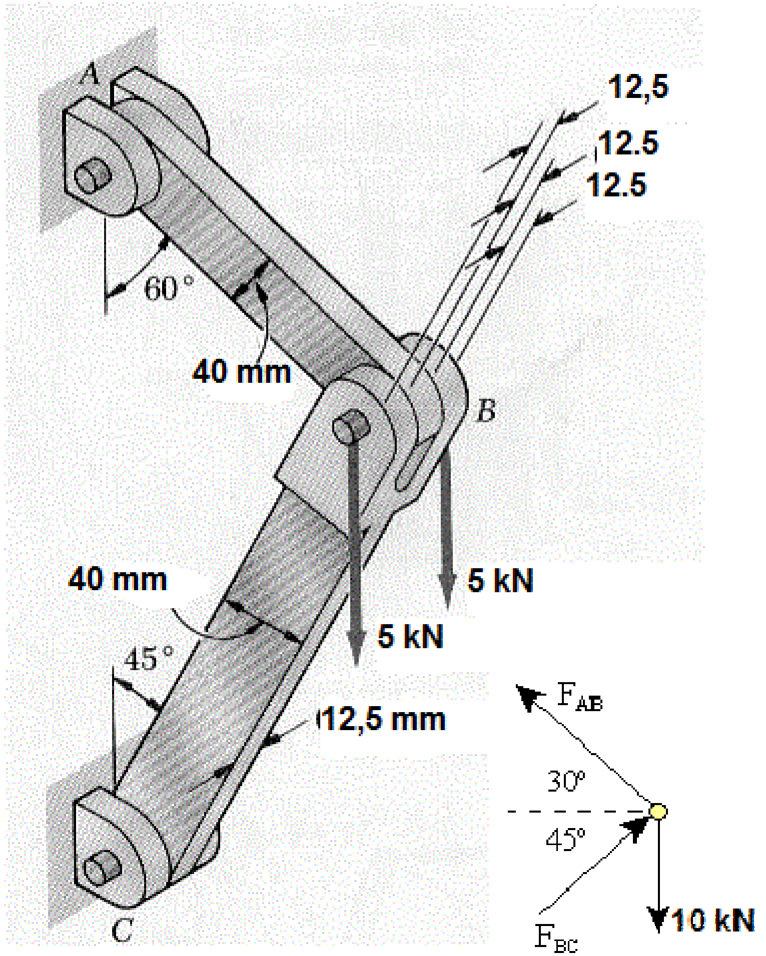
$$\sigma_b = \frac{P}{A_b}$$

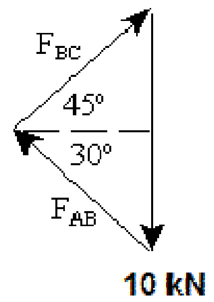


Příklad

Two vertical forces - each 5 kN, act on the pin B of the structure. At places A, B and C there are pins with a diameter of 16 mm
Specify the maximum value of the normal voltage in the AB and BC bars
Determine shear stresses in each pin
Determine the pressures in the punches for fingerprint control

- Answer
- Normal stresses
- F(AB) = 7,33 kN tah, 14,7 MPa
- F(BC) = 8,96 kN tlak; -17,9 Mpa
- Shear stresses/contact pressure
- Čep A : 18,2 / 36,7 MPa
- Čep B : 24,8 / 50 MPa
- Čep C : 22,3 / 44,8 Mpa





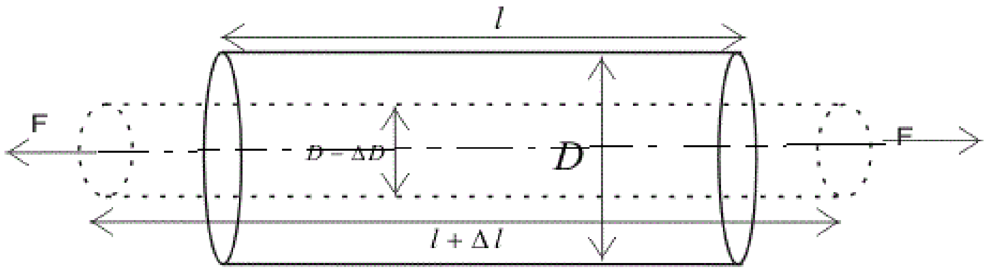
Transverse deformation of a rod loaded by a tensile force
Poisson's number

- The rod loaded by the tension is extended, but it also changes its lateral dimensions - it narrows

The transverse proportional strain ΔD / D is in absolute value μ times smaller than the relative elongation ε = Δl / l

μ is the Poisson number

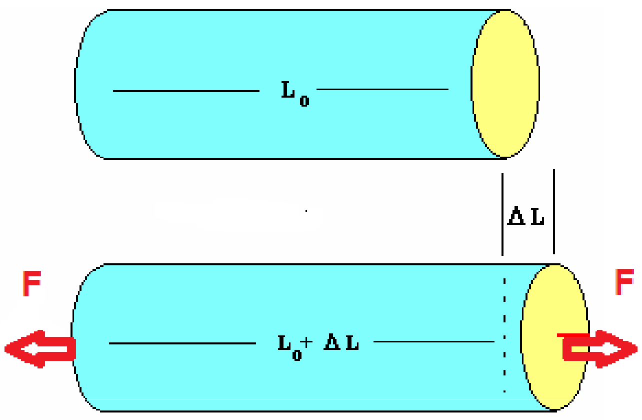
Young's elastic modulus E and Poisson number μ are two independent material constants of most structural isotropic materials



látka	E*10^{-5} [MPa]	G*10^{-5} [MPa]	μ
brass	0,99	0,365	0,36
steel	2,00	0,810	0,29
lead	0,16	0,056	0,44
aluminium	0,71	0,264	0,34
copper	1,23	0,455	0,35
platinum	1,70	0,610	0,39
argent	0,79	0,287	0,37
zinc	0,90	0,360	0,25
glass	0,6 až 0,7	0,26 až 0,32	0,2 až 0,27

$$\varepsilon = \frac{\Delta l}{l}, \quad \frac{\Delta D}{D} = -\mu\varepsilon, \quad \varepsilon = \frac{\sigma}{E}, \sigma = \frac{F}{S}, S = \frac{\pi D^2}{4}, \varepsilon = \frac{F}{ES}, \quad \Delta D = -\mu\frac{FD}{ES}$$

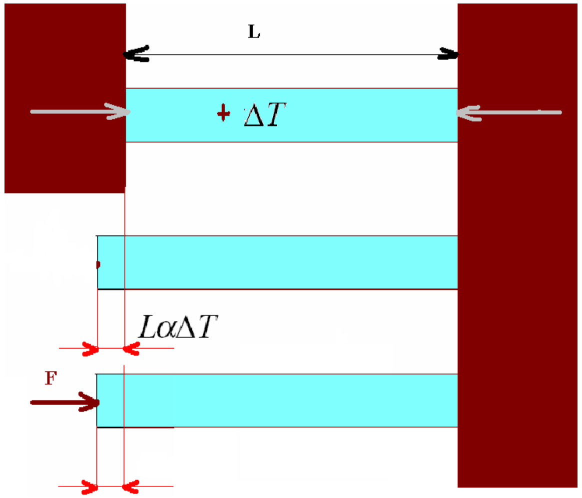
Temperature deformations and stresses



The rod is loaded with the force F and at the same time it is heated by ΔT

$$\varepsilon_T = \alpha \Delta T, \quad \varepsilon_\sigma = \frac{\sigma}{E}, \quad \varepsilon = \varepsilon_\sigma + \varepsilon_T$$

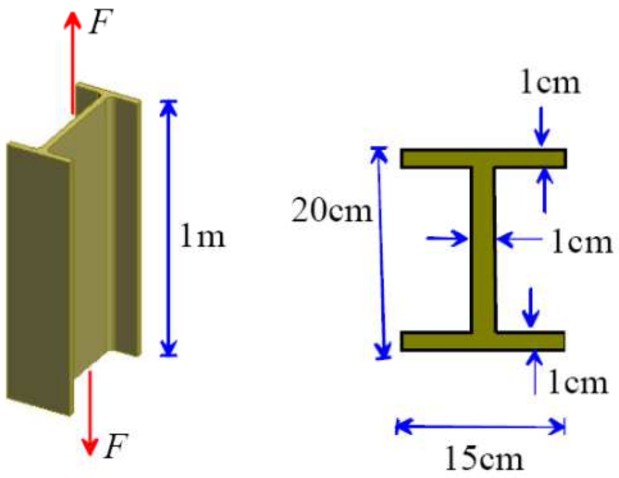
$$\varepsilon = \alpha \Delta T + \frac{F}{ES}, \quad \Delta L = \varepsilon L$$



If we prevent thermal expansion, a compressive force and thus a compressive stress is produced in the rod

$$\Delta L = 0 \Rightarrow \varepsilon = 0 \Rightarrow \alpha \Delta T - \frac{F}{ES} = 0 \Rightarrow \sigma = -E\alpha \Delta T$$

Problem



•the steel rod in the picture should be loaded with tensile force.

? the maximum force the rod can transfer without permanent deformations

? the maximum force that the rod will transmit without breaking

? the safety force n = 1.5 with respect to the yield and the elongation of the rod at this force

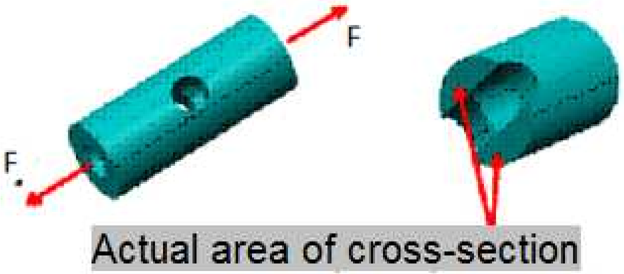
? rod extension when heating at 20 ° C

? tension in the rod if the thermal elongation is avoided

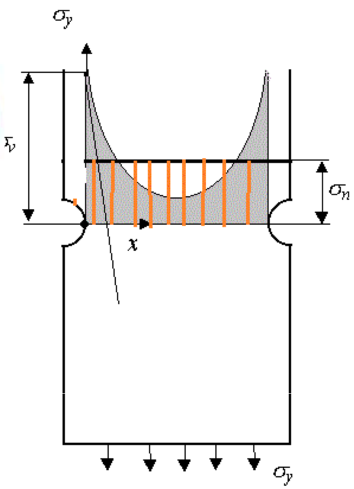
Given: E = 196GPa, σk = 220MPa,
σpt = 430 MPa, α = 12 * 10-6 K-1

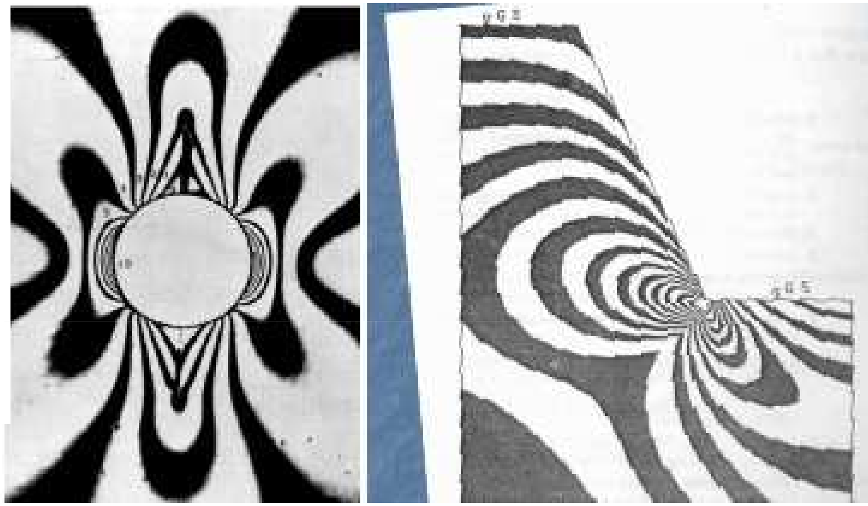
Odpověď: 1,05/2/0,7 MN; 0,75/0,24 mm; 47MPa

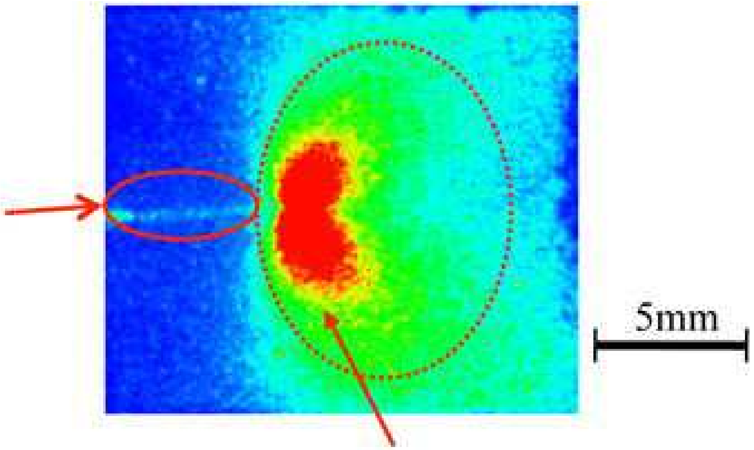
Stress concentrations around shape changes and holes

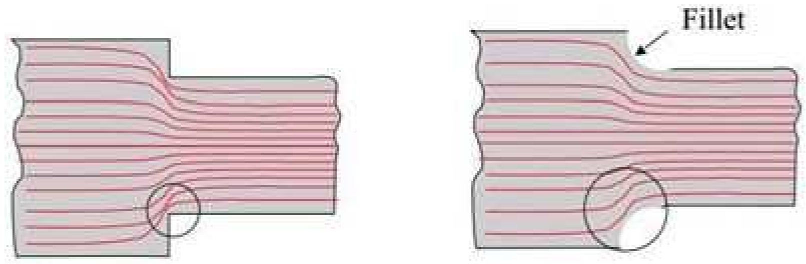


$$\sigma_{max} \doteq 3\frac{F}{S}$$

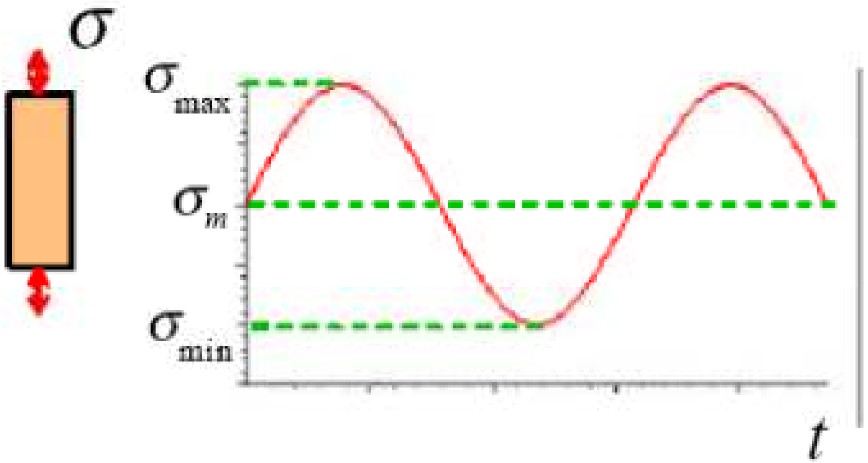








Fatigue limit - cyclic load



- The fatigue limit of the material is determined experimentally by an controlled fatigue test - usually a bending in rotation or cyclical tension.

  Stress varies with time according to relationship

- $\sigma_a(t)= \sigma_0 \sin(2\pi t/T_0)$

- $\sigma_a(t)$ is the *amplitude* of stress and T_0 is the *period* – cycle time.

- It is relatively easy to design components subjected to static loads (design of a safe load relative to the yield strength or to the strength).

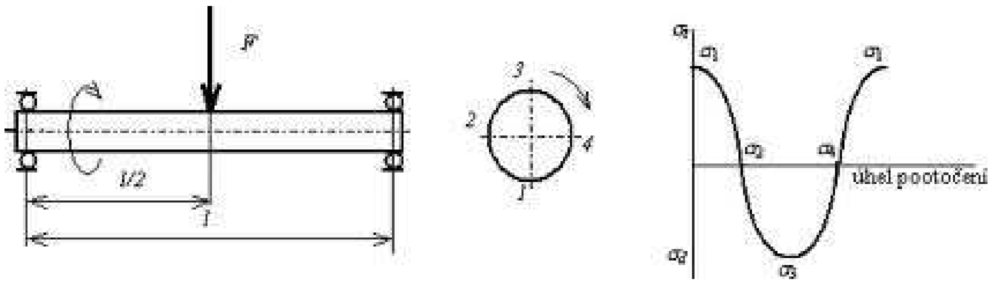
- However, most parts are broken down under dynamic loads.

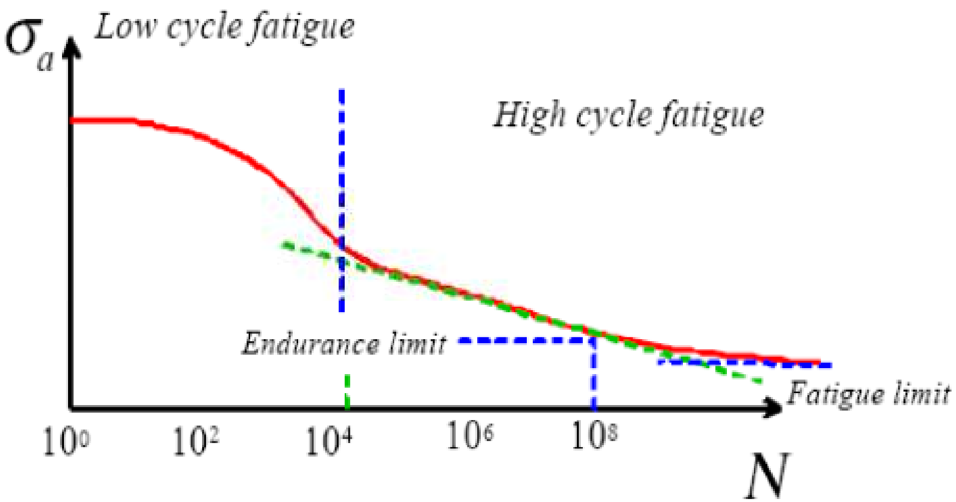
- If a component is loaded with a cyclically varying force, it is damaged by material fatigue - the part is working for a long time and is suddenly broken.

- Fatigue is caused by the formation and growth of cracks in the material due to cyclic loading.

- The breach occurs when the cracks become critical.

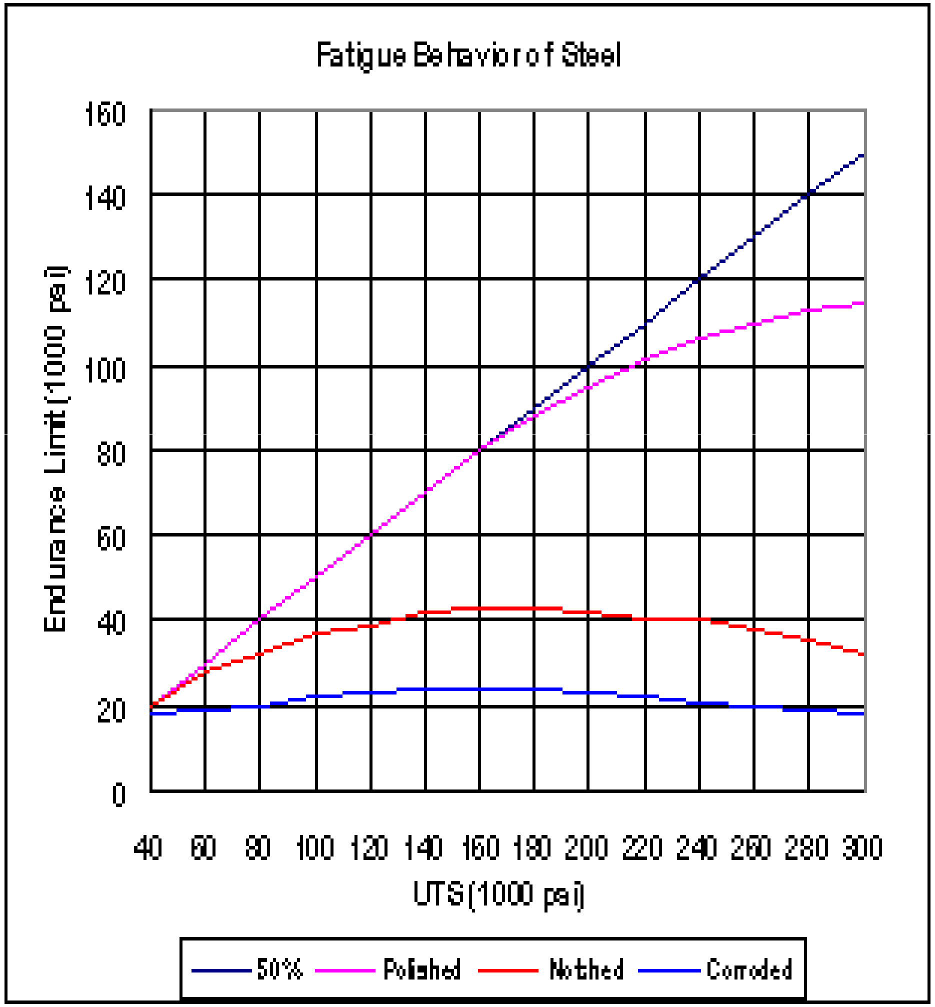
Fatigue limit





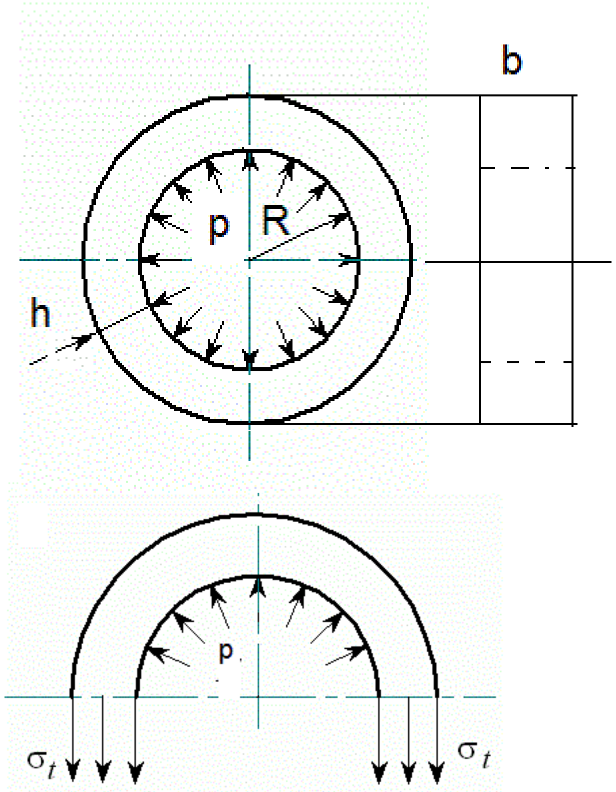
- You need to know how long a part under the cyclical load will endure
- Experiments show that the material breaks down after a certain number of Nf cycles.
- The number of cycles depends on the stress amplitude σa.
- The relationship between σa and N expresses the so-called "S-N curve" of the material.
- At high stress, the material breaks rapidly - between 1 and 1000 cycles
- At a lower stress in the material, it can withstand more than 10,000 to 10,000,000 cycles
- Some materials have a **fatigue limit**
  - if the stress amplitude is lower than the fatigue limit, the component will not break through the "infinite" number of cycles
- The fatigue limit is usually defined as the amplitude of the stress at which the material endures more than 10^8 load cycles

Fatigue limit



- A simple rule states that for steel with a tensile strength less than 1000 MPa, the fatigue limit is approximately 45-50% when the surface of the test specimen is smooth and polished (pink curve).
- The graph labeled "Notched" shows a dramatic reduction in fatigue load due to the stress concentration at the point of sudden shape change (recesses, grooves, sharp shoulders, and transitions)
- The surface of the component has a primary influence on the fatigue limit (surface cracks). It is evident from the "Corroded" blue curve. “
- Fatigue cracks usually run out of existing surface cracks.

Thin ring loaded by pressure



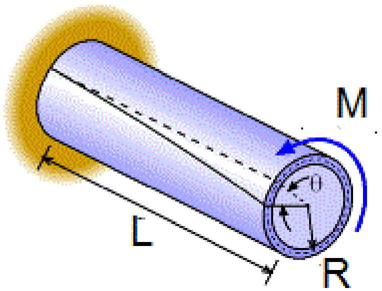
A thin ring whose thickness h = 5mm is much smaller than the radius R = 60mm (at least 10x) is loaded with radial pressure p = 6MPa. The ring width is b = 10mm. We determine the tension in the ring and its expansion ΔR.
E = 2 * 105 Mpa.
We divide the ring with the imaginary section by the symmetry plane in two parts. We assume that the normal circumferential stress in the imaginary section is evenly distributed (it is constant). From the equilibrium condition to the axis direction we have:
:

$$\sigma_t\, 2bh = pb2R \Rightarrow \sigma_t = p\frac{R}{h} \Rightarrow \varepsilon_t = \frac{\sigma_t}{E}$$

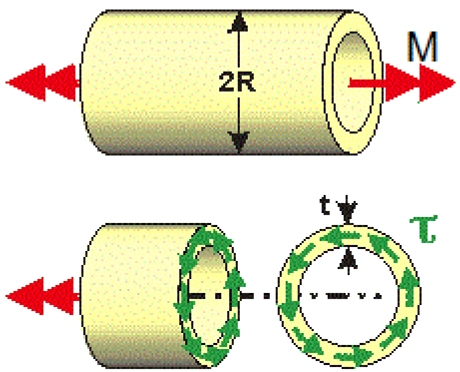
$$\sigma_t = 72MPa, \quad \Delta R = 0{,}0216mm$$

$$\varepsilon_t = \frac{\Delta O}{O} = \frac{2\pi\left(R+\Delta R\right)-2\pi R}{2\pi R} = \frac{\Delta R}{R} \Rightarrow \Delta R = \frac{p}{E}\frac{R^2}{h}$$

Torsion of thin-walled tubes



A long thin wall tube of diameter D = 2R and wall thickness t is stressed by the torque M acting in the plane of the cross section around the pipe axis. The pipe is twisted and the shear stress occurs in the cross section. Because the tube is thin, we can predict that the shear stress is constant. The ends of the tube are rotated with respect to the angle θ.



$$M = 2\pi R^2 t\,\tau, \quad W_k = 2\pi R^2 t, \quad \tau = \frac{M}{W_k}$$

$$\theta = \vartheta L, \ \vartheta = \frac{\gamma}{R}, \ \gamma = \frac{\tau}{G}, \quad G = \frac{E}{2(1+\mu)}$$

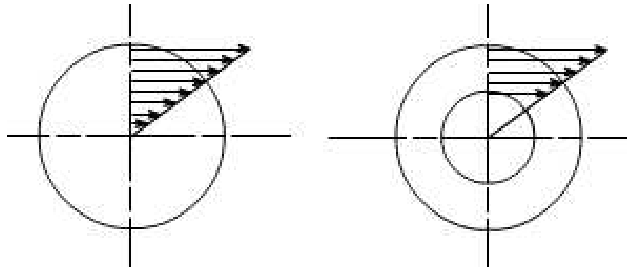
$$\theta = \frac{ML}{G 2\pi R^3 t}, \quad J_p = 2\pi R^3 t$$

$\tau = shear\ stress, W_k =$ modulus of cross-section in torsion, $\theta =$ angle of twist, γ=shear strain,
$\vartheta = \theta/\mathrm{L}$, G=modulus elasticity in shear, $J_p =$ quadratic polar moment of cross-section

Torsion of shafts - example

Circular cross-sect.: $W_k = \frac{\pi d^3}{16},\ J_p = \frac{\pi d^4}{32}.$

Ring c-s: $J_p = \frac{\pi\left(D^4 - d^4\right)}{32}, W_k = \frac{\pi D^3}{16}\left(1 - \left(\frac{d}{D}\right)^4\right).$

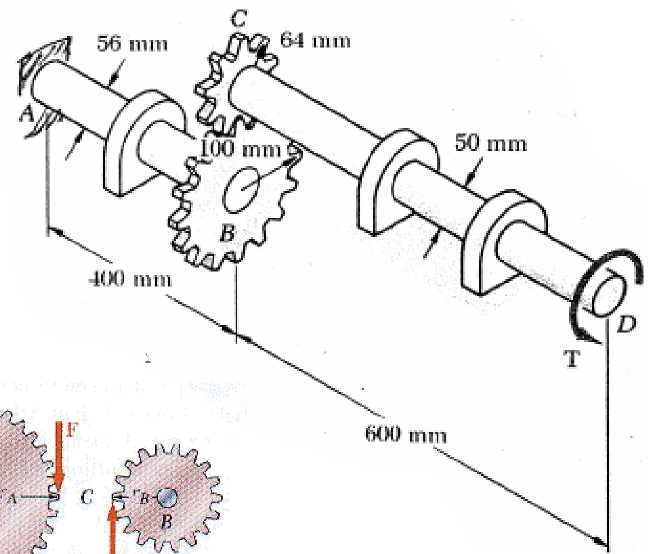


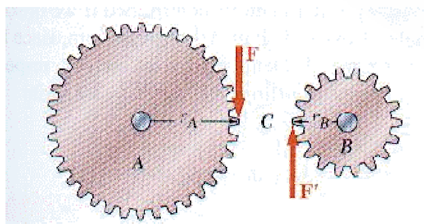
The shear stress in the full cross section is not constant but changes linearly depending on the radius. It is largest at the outer edge of the cross section.

$$\tau(r) = \frac{M_k}{J_p} r$$

Example:

The two shafts transmit torque by gears B and C. The torque T applied at point D is T = 900 Nm. Determine the maximum shear stresses in the shafts and the angle of twist between the ends C and D. The modulus of shear is G=8*10^{4}MPa



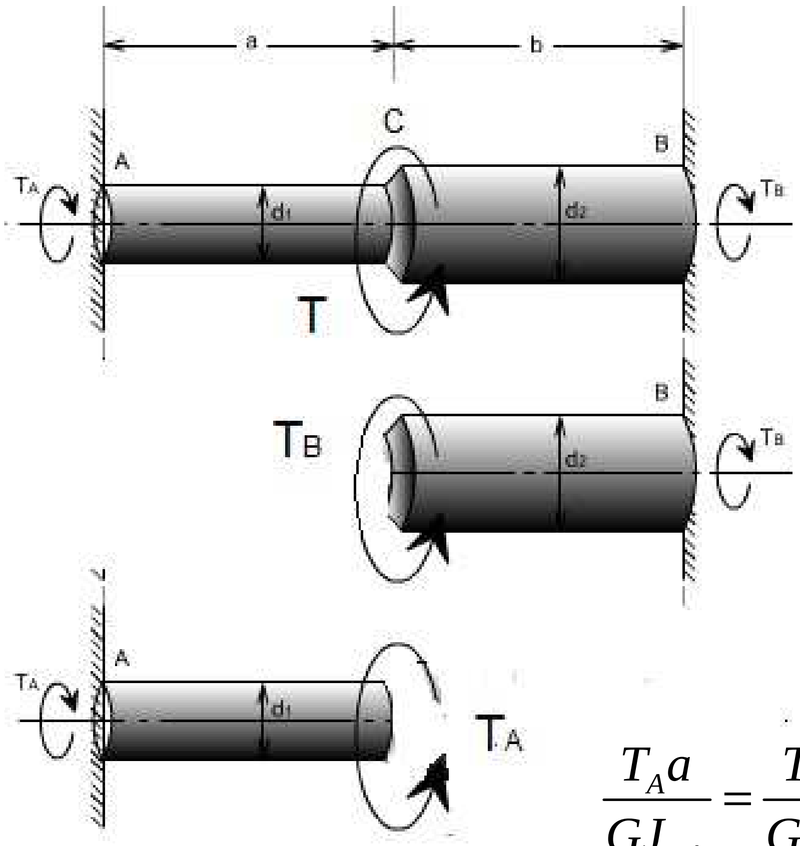


Maximum shear stress in C-D: $\tau_{CD} = \frac{T}{W_{kCD}} = 36,7 MPa$

Angle of twist C-D: $\theta_{CD} = \frac{TL_{CD}}{GJ_{pCD}} = 0,011 rad$

Stress in shaft A-B: $\tau_{AB} = \frac{T\frac{100}{64}}{W_{kAB}} = 40,8 MPa$

Statically indeterminate torsion - example



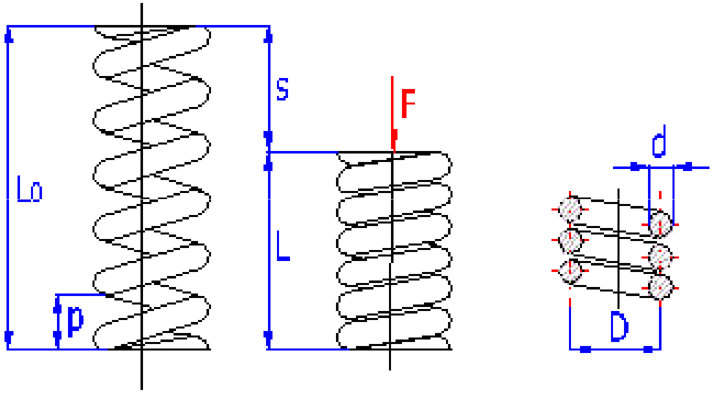
The rod is loaded with a torque T = 1 kNm at the C point and is built in at both ends so that the cross sections A and B can not rotate with respect to each other $\theta_{AB} = 0$. What is the stress in each part of the rod. Given: a = b = 250mm, d1 = 25mm, d2 = 40mm, G = 8 * 10^4MPa. Moment T is divided into both parts of the rod - the stiffer part of the rod will carry most of the torque. The equilibrium condition T = TA + TB applies, at the same time the cross section C must be equal for both the left and right bars: θCA = θCB

$$\frac{T_A a}{GJ_{pA}} = \frac{T_B b}{GJ_{pB}} \Rightarrow T_A = \frac{b}{a}\frac{J_{pA}}{J_{pB}} T_B = \frac{b}{a}\left(\frac{d_1}{d_2}\right)^4 T_B$$

$$M_A = 0{,}132 kNm,\ \tau_A = 43 MPa,\ M_B = 0{,}868 kNm,\ \tau_B = 69 MPa.$$

Compact wound cylindrical springs

The spring diameter is D, the wire diameter d, the number of threads is n, the shear modulus is G. The wire of spring is loaded by torque $M_k = F*D/2$, maximum shear stress in cross-section of wire:



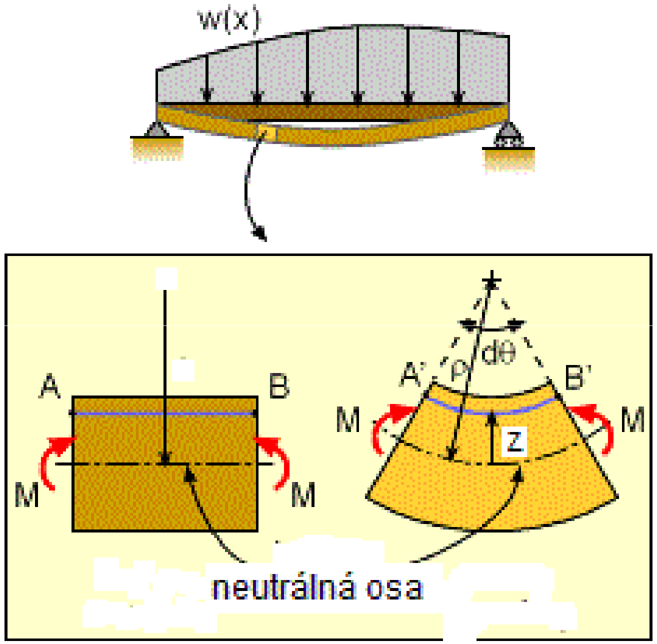
$$\tau_{\max} = \frac{F\frac{D}{2}}{W_k} = \frac{F\frac{D}{2}}{\frac{\pi d^3}{16}}, \text{ compression } s = \frac{8FnD^3}{Gd^4}.$$

Příklad:

Determine the maximum shear stress and the elongation of the helical spring. F=1,1 kN, D=200mm,d=20mm, n=20, G=8,4*10^4 Mpa

Ans.: 80,1MPa, 104,8mm

Stress in beams in bending



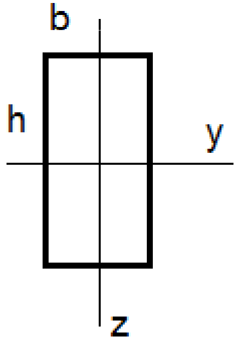
When the beam is bend, we assume that the cross-sections of the beam do not deplanate = they remain plane and they only incline. Bernoulli's hypothesis: There is a neutral axis in the bent bar, which does not extend or shorten and the cross-sections after deformation remain perpendicular to it. The neutral axis (NA) passes through the centroid of the cross section. Fibers below and above NA are either elongated or shortened in course of the beam deformation, their strain is:

$$\varepsilon = \frac{A'B' - AB}{AB} = \frac{(\rho - z)\,d\theta - \rho\,d\theta}{\rho\,d\theta} = -\frac{z}{\rho} \Rightarrow \sigma = E\varepsilon = -E\frac{z}{\rho}$$

$$\text{curvature: } \frac{1}{\rho} = \frac{M_o}{EJ_y} \Rightarrow \sigma = -\frac{M_o}{J_y}z, \quad \sigma_{\max} = \frac{M_o}{W_o}$$

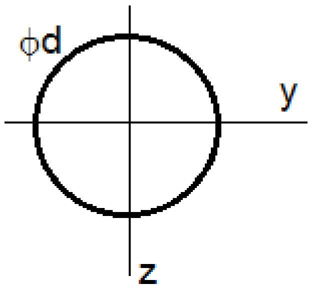
Cross-sectional values

- Quadratic moments of cross-sectional area and modules in torsion or bending, radii of inertia, etc. of commonly used cross-sectional shapes are in the tables.
  Let's just say two of them:



Rectangular area:

$$J_y = \frac{bh^3}{12}, \quad W_{oy} = \frac{bh^2}{6}, \quad J_z = \frac{b^3h}{12}, \quad W_{oz} = \frac{b^2h}{6}.$$

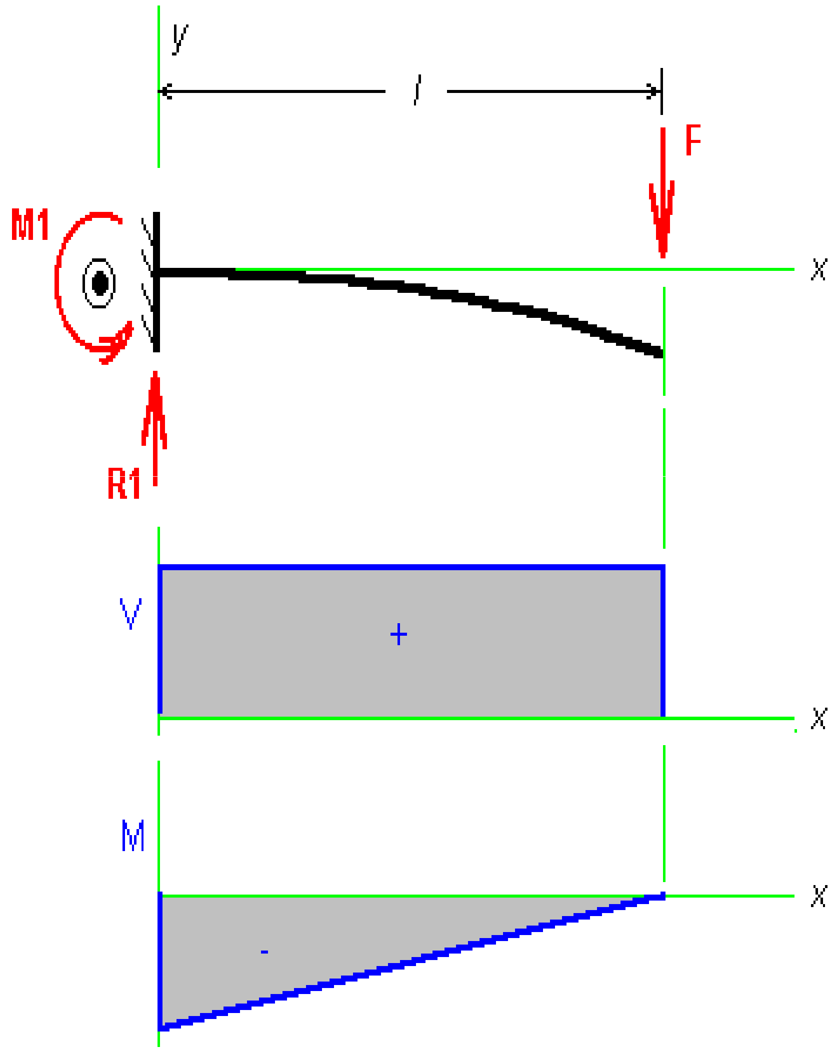


Circular area:

$$J_y = J_z = \frac{\pi d^4}{64}, J_p = J_y + J_z = \frac{\pi d^4}{32}, W_o = \frac{\pi d^3}{32}, W_k = \frac{\pi d^3}{16}.$$

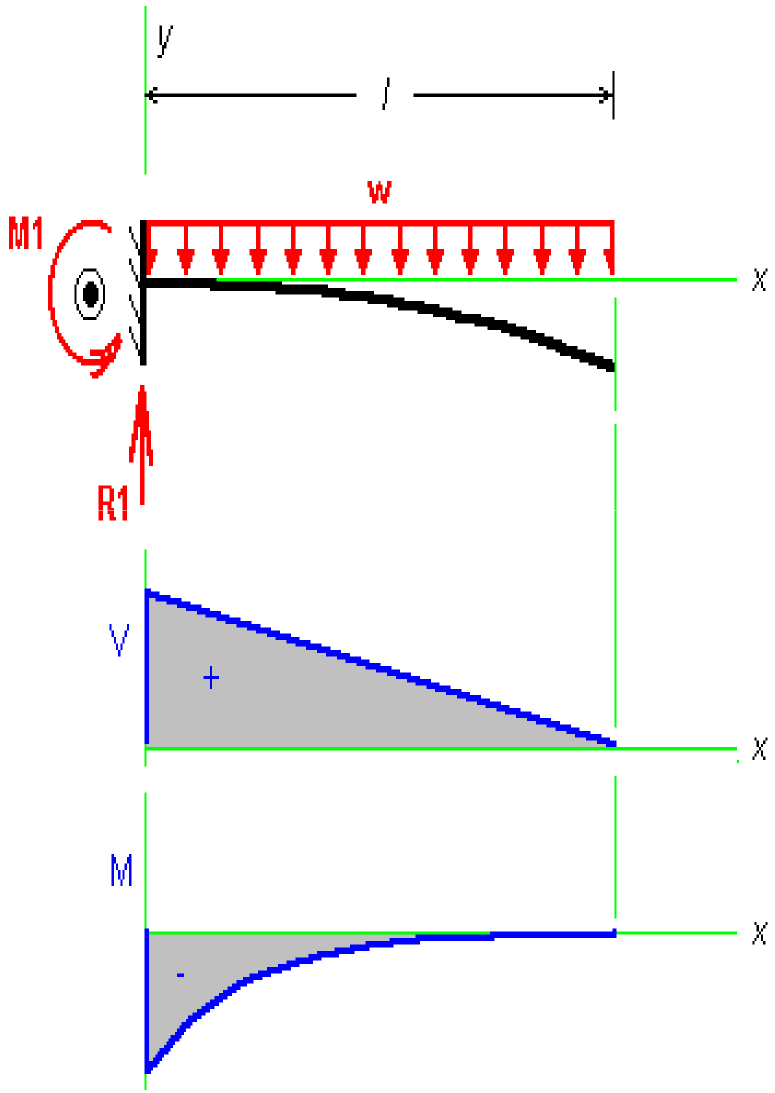
Built in beams

- Beam built in at the left end is loaded by single force at the other end
- Reaction
- $R_1 = F$
- $M_1 = -F\,L$ (maximum bending moment)
- Shear force V= const
- Inner bending moment is linear
- Maximum deflection at x = L
- $y_{max} = (F\,L^3)/(3\,E\,I)$



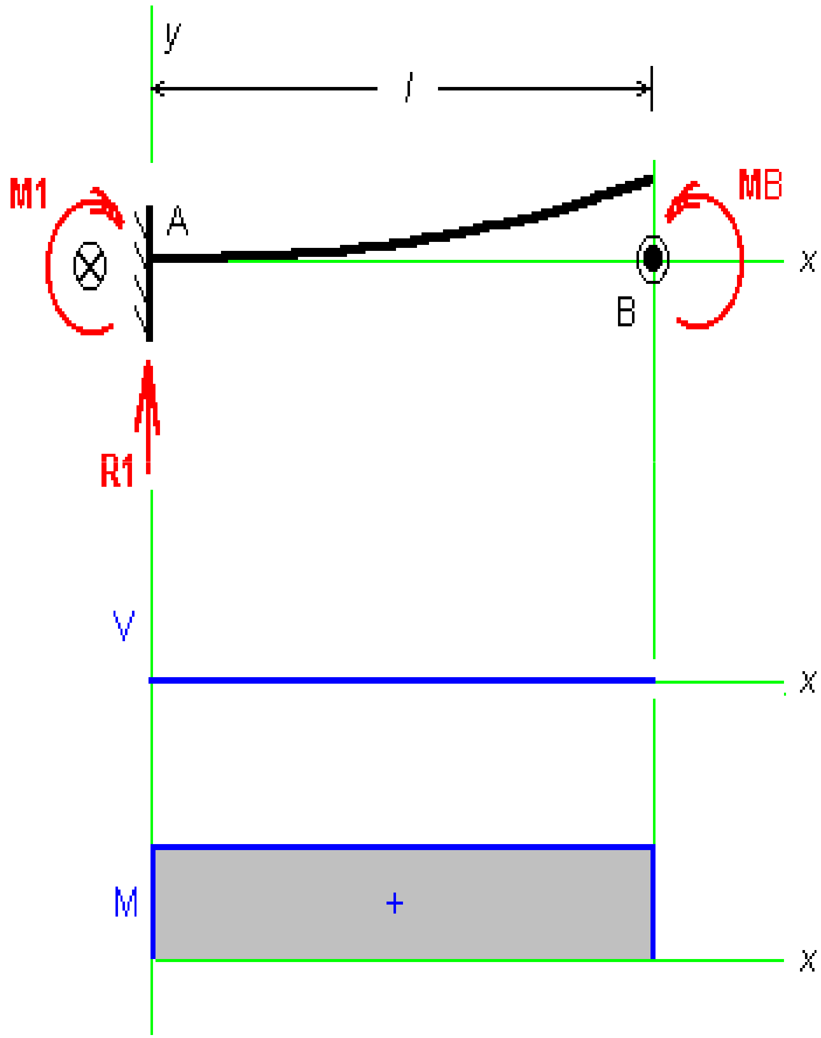
Built in beam

- Beam built in at the left end is loaded by uniformly distributed load w [N/m]
- Reactions
- $R_1 = w L$
- $M_1 = -(w L^2)/2$
- (maximum bending moment)
- Shear force V is linear
- Moment is a parabola of 2nd degree.
- Maximum deflection at x = L
- $y_{max} = (w L^4) /(8EI)$



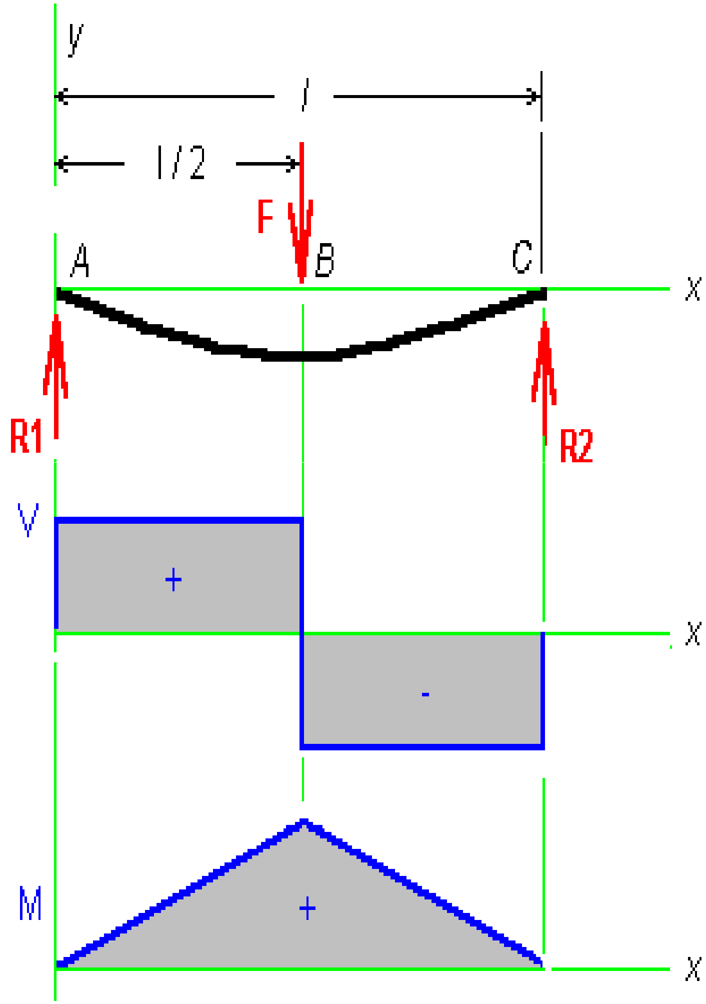
Built in beam

- Beam built in at the left end is loaded by single moment M at the other end
- Reactions
- $R_1 = 0$
- $M_1 = M_B$ (maximum bending moment)
- Shear force V= 0
- Moment is const
- Maximum deflection at x = L
- $y_{max} = -(M_B L^2) / (2 E I)$



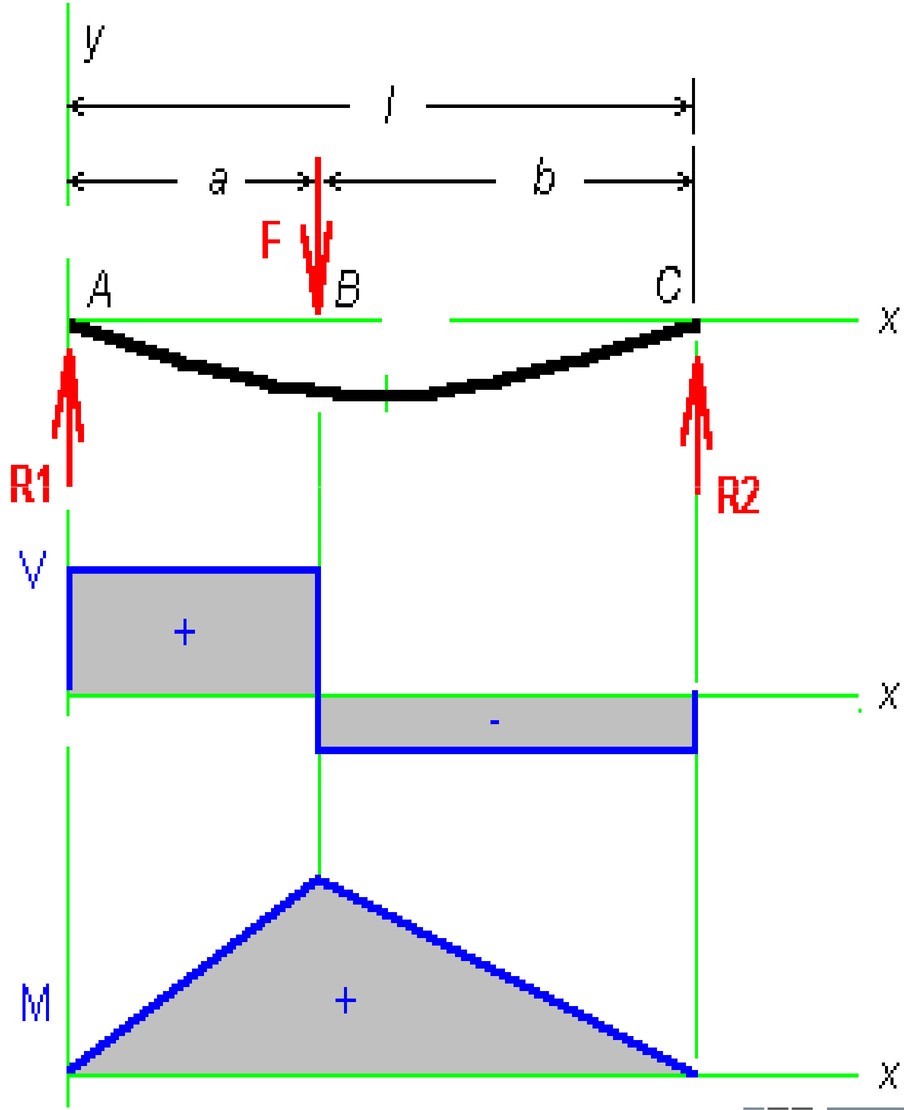
Simply supported beams

- Beam on two supports is loaded by
- A single force F in the middle
- Reactions $R_1 = R_2 = F / 2$
- Maximum bending moment is in the middle under the loading force
  $M_{B=}$ F L / 4
- Shear force is constant V=F/2 in the first part of beam and V=-F/2 in the second part
- Moment is linear function of x
- Maximum deflrction at x = L/2
- $y_{max} = (F L^3)/(48 E I)$



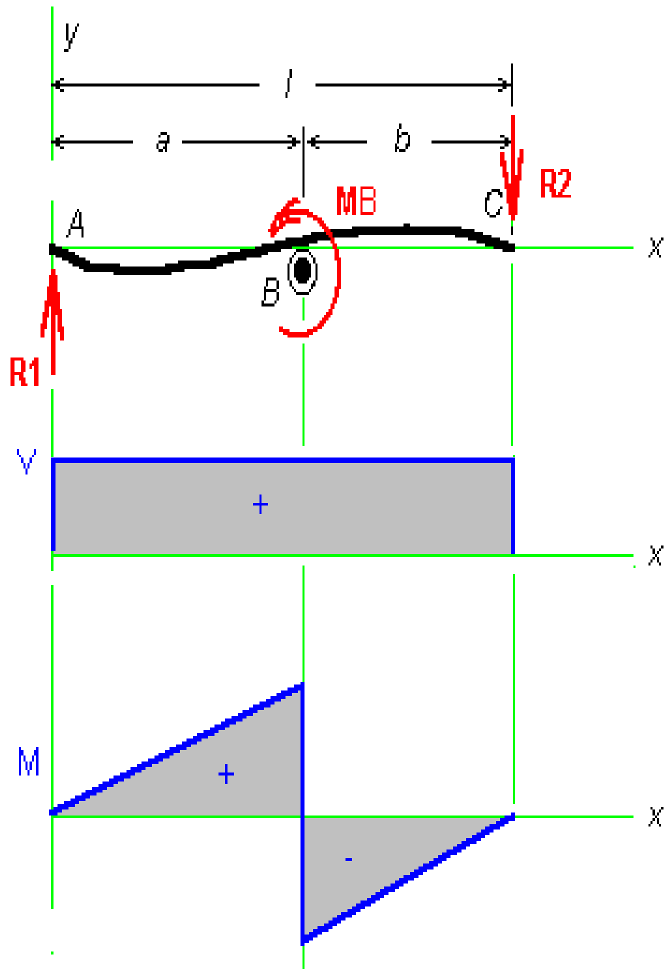
Simply supported beam

- Beam on two simple support is loaded by a single force F
- Reactions R_1 = F b / L, R_2 = F a / L
- Max bending moment is under the force F
- M_B = F a b / L
- Shear force V is constant V= R_1 in the first part and V= - R_2 in the second part of beam
- Moment is a linear function of x
- The deflection in the point x = L/2
- $y(L/2) = (F\ aL^2)[3/4-(a/L)^2]/(12E\ I)$



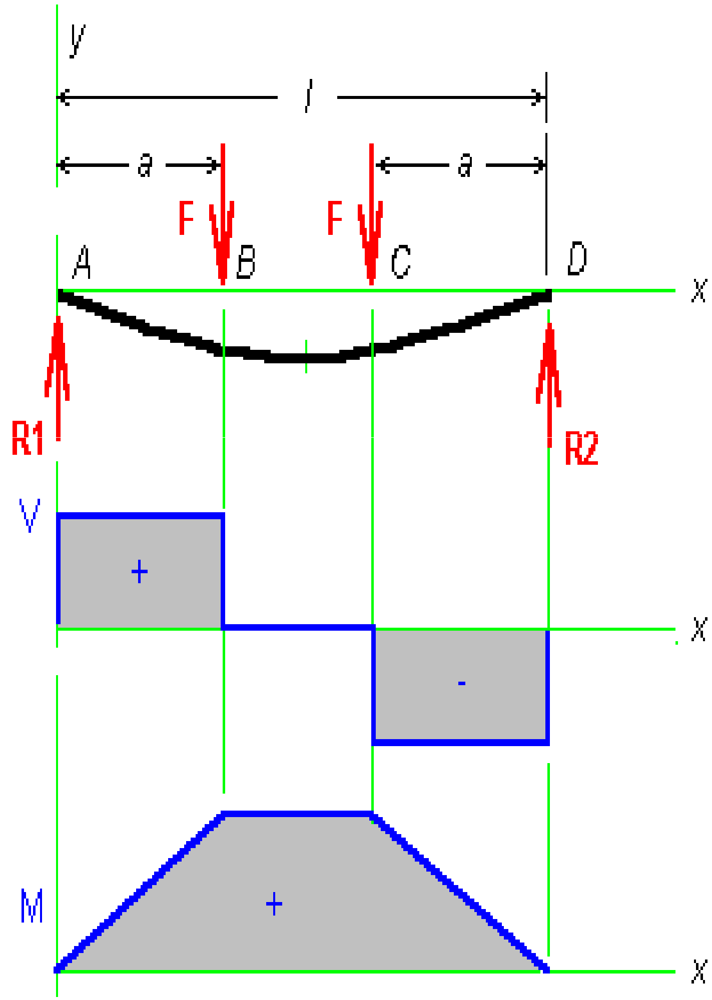
Simply supported beam

- The simply supported beam is loaded by a single bendin moment M_B
- Reactions $R_1 = -R_2 = M_B / L$
- Internal bending moment changes in the point B, it is different from right and left
- $M_L = M_B a / L$, $M_P = -M_B b / L$
- Shear force V is constant $V = R_1$
- Moment is linear function of x



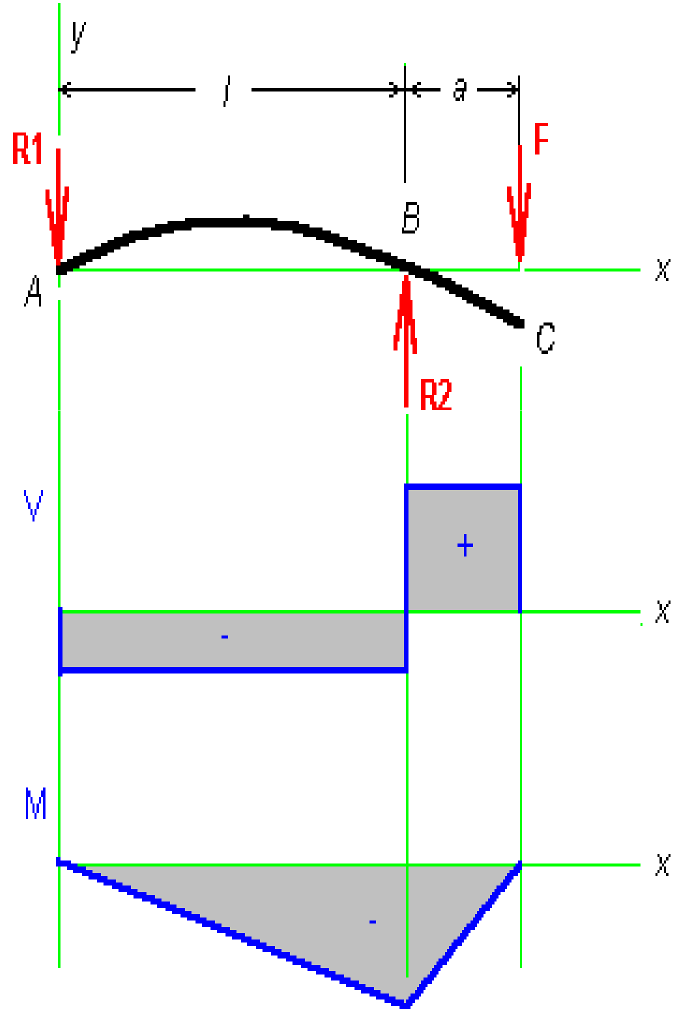
Simply supported beam

- The simply supported beam is loaded symmetrically by the two forces F
- Reactions $R_1 = R_2 = F$
- Maximum bending moment is in the middle of beam M_{max}=F a
- Shear force V is constant V=F in the first part of beam and V=-F in the second part. It is zero in the middle of beam.
- Moment is the linear function of x
- Maximum deflection in x = L/2
- $y_{max} = (F\ a)\ (4\ a^2 - 3\ L^2)\ /(24\ E\ I)$

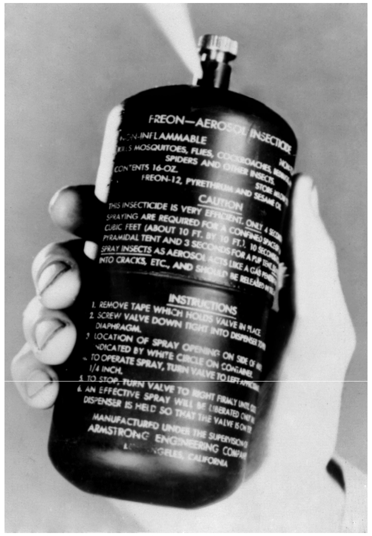


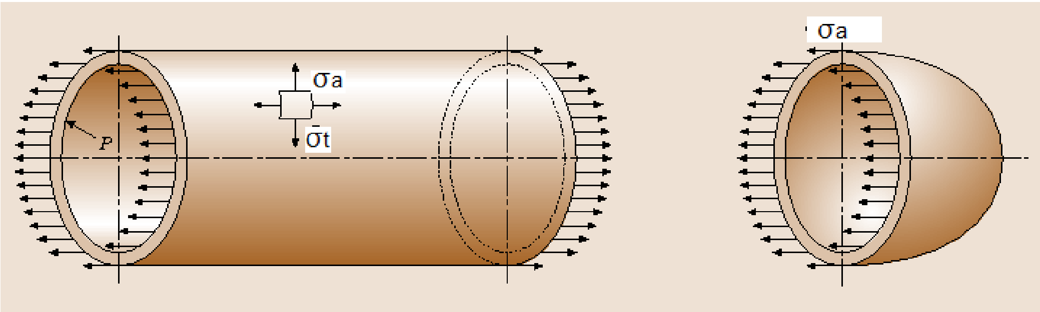
Simply supported beam with an overhanging end

- The beam on the two supports with the overhang is loaded with simple force F
- Reations R_1 = -F a / L , R_2 = F (L + a) / L
- Max bending moment is in the point B
  $M_B = M_{MAX} = - F a$
- Shear force V is constant V=- R_1 in the first part and V=F in the second part of beam
- Moment is athe linear function of x
- Maximum deflection at point C
- $y_{max} = y_C = Fa^2L(1+a/L)/(3EI)$



Plain stress- Thin-walled pressure vessel





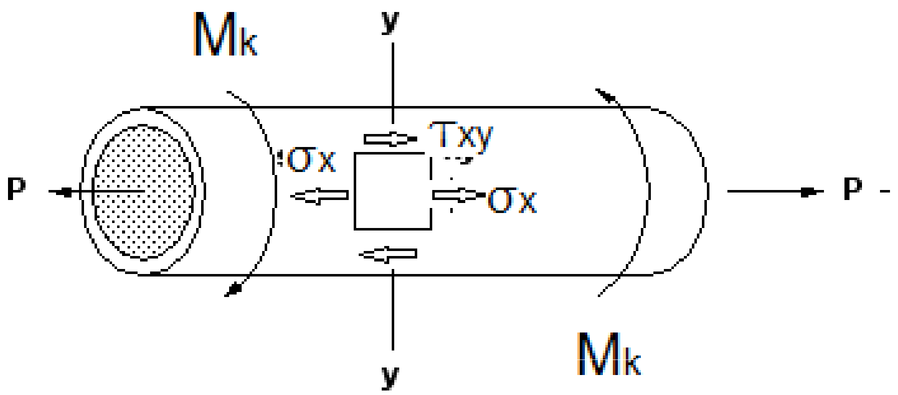
In addition to the circumferential stress (see thin-walled ring) there is the axial stress which is constant in the thickness of the wall and is determined by the condition of the equilibrium of the bottom separated by an imaginary section :

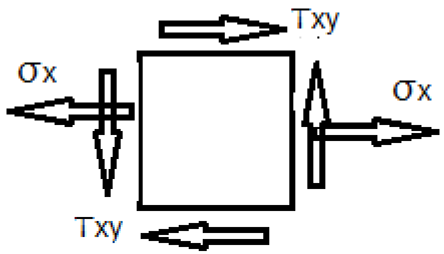
$$\sigma_t = p\frac{R}{h}, \ \sigma_a 2\pi R h = p\pi R^2 \Rightarrow \sigma_a = \sigma_t/2$$

Strain in circumferential and axial direction from Hooke's law:

$$\varepsilon_t = \frac{1}{E}(\sigma_t - \mu\sigma_a), \varepsilon_a = \frac{1}{E}(\sigma_a - \mu\sigma_t), \Delta R = \varepsilon_t R$$

Thin tube loaded in tension and torsion





The plain stress - a combination of normal tensile stress and shear stress
Strength is characterized by an equivalent stress:

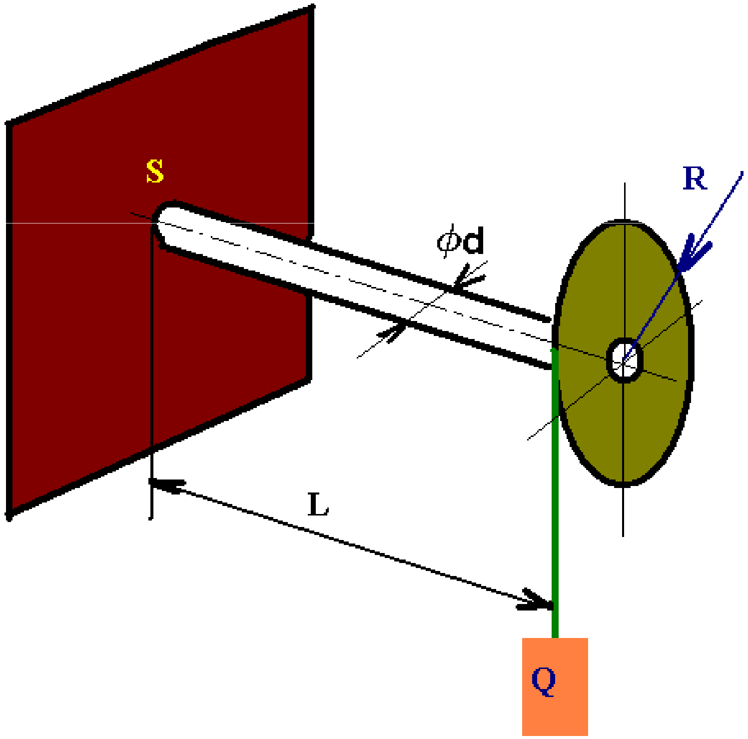
$$\sigma_x = \frac{P}{S} = \frac{P}{2\pi Rh}, \tau_{xy} = \frac{M_k}{W_k} = \frac{M_k}{2\pi R^2 h}, \ \sigma_{ekv} = \sqrt{\sigma_x{}^2 + 3\tau_{xy}{}^2}$$

Stresses in a shaft

- Maximal bending moment in point S
- M_{bend} = Q L
- Constant twist moment
- M_{twist}= Q R
- Combination of normal and shear stresses:

$$\sigma_o = \frac{QL}{W_o} = \frac{QL}{\pi d^3/32}, \tau_k = \frac{QR}{W_k} = \frac{QR}{\pi d^3/16},$$

$$\sigma_{ekv} = \sqrt{\sigma_o^{\,2} + 3\tau_k^{\,2}}$$



Recommended

- T.E. Pilpot, Mechanics of materials, Mecmovie http://web.mst.edu/~mecmovie/
- MATHalino.com http://www.mathalino.com/reviewer/mechanics-and-strength-of-materials/mechanics-and-strength-of-materials

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