Response to Nonharmonic Forces

Iva Petríková

Dept. of Applied Mechanics

evropský sociální fond v ČR
EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
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Introduction

- Response of single degree of freedom system to more general forcing functions
- Forcing function F(t) to be periodic if there exists *real number T (period)* such that $F(t+T)=F(T)$, nT is also period,
- Periodic function can be written as the sum of harmonic functions using Fourier series.

Content

- Fourier Series
- Response to polyharmonic function
- Periodic forcing functions
- Response to Impulsive motion
- Example – Response to Rectangular Forcing Function

Fourier Series and Determination of the Fourier Coefficients

$$F(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\omega t+\sum_{n=1}^{\infty}b_n\sin n\omega t \qquad F(t)=F_0+\sum_{n=1}^{\infty}F_n\sin(n\omega t+\varphi_n) \qquad \omega=\frac{2\pi}{T}$$

Coefficients a_0, a_n, b_n

$$a_n=\frac{2}{T}\int_0^T F(t)\cos(n\omega t)\,\mathrm{d}t$$ for n= 0,1,2…

$$b_n=\frac{2}{T}\int_0^T F(t)\sin(n\omega t)\,\mathrm{d}t$$ for n= 1,2…

Relations for a_n , b_n , F_n , φ_n

$$F_0=\frac{a_0}{2} \qquad F_n=\sqrt{a_n^2+b_n^2} \qquad \varphi_n=\operatorname{atan}\left(\frac{a_n}{b_n}\right)$$

Response to polyharmonic function

$$m\ddot{x} + b\dot{x} + kx = \sum_{n=1}^{N} F_n \sin\left(\omega_n t + \varphi_n\right)$$

The principle of superposition can be applied for linear system, particular solution is polyharmonic function, steady-state response is given by the sum of responses on components of harmonic functions

$$x\left(t\right) = x_h + x_p = x_h + \sum_{n=1}^{N} x_{pn}$$

$$x_p\left(t\right) = \sum_{n=1}^{N} x_n \sin\left(\omega_n t + \vartheta_n\right)$$

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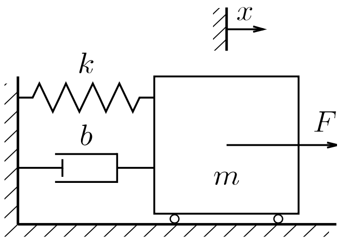
Vibration of the single DOF system under periodic f.f.

- Differential equation od single DOF system



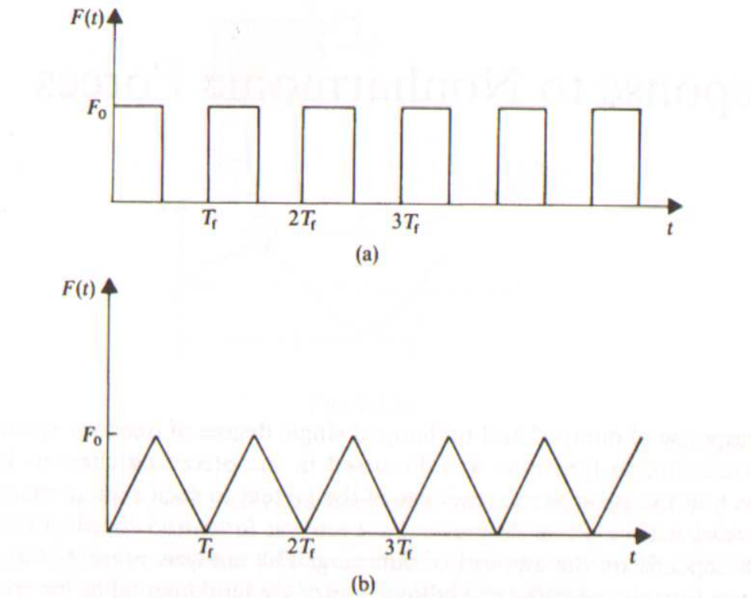
$$m\ddot{x} + b\dot{x} + kx = F(t)$$

- *F(t)* is expressed in terms of Fourier series

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$F(t) = F_0 + \sum_{n=1}^{\infty} F_n \sin(n\omega t + \varphi_n)$$

$$m\ddot{x} + b\dot{x} + kx = F_0 + \sum_{n=1}^{\infty} F_n \sin(\omega_n t + \varphi_n) \qquad \omega_n = n\omega$$



Examples of periodic forcing functions (Shabana,1997)

Response to the periodic forcing function

$$m\ddot{x} + b\dot{x} + kx = F_0 + \sum_{n=1}^{\infty} F_n \sin(\omega_n t + \varphi_n)$$

The principle of superposition can be applied to obtain the paricular solution x_p :

1) Response to the constant term F_0:

$$m\ddot{x}_{p0} + b\dot{x}_{p0} + kx_{p0} = F_0$$

$$x_{p0} = C$$

$$\dot{x}_{p0} = \ddot{x}_{p0} = 0$$

$$kC = F_0$$

$$x_{p0} = C = \frac{F_0}{k}$$

2) Response due to each of the terms:

$$F_n \sin(\omega_n t + \varphi_n)$$

$$x_{pn} = \frac{F_n / k}{\sqrt{\left(1-\eta_n^2\right)^2 + \left(2\zeta\eta_n\right)^2}} \sin(\omega_n t + \varphi_n - \psi_n)$$

$$\eta_n = \frac{\omega_n}{\Omega} = \frac{n\omega}{\Omega} = n\eta \quad \Omega = \sqrt{\frac{k}{m}}$$

Response to the periodic forcing function

$$\psi_n = \operatorname{arctg}\left(\frac{2\zeta\eta_n}{1-\eta_n^2}\right)$$

Complete solution

$$x(t) = x_h + x_p$$

$$x_p = x_{p0} + \sum_{n=1}^{\infty} x_{pn}$$

$$= \frac{F_0}{k} + \sum_{n=1}^{\infty} \frac{F_n / k}{\sqrt{\left(1-\eta_n^2\right)^2 + \left(2\zeta\eta_n\right)^2}} \sin\left(\omega_n t + \varphi_n - \psi_n\right)$$

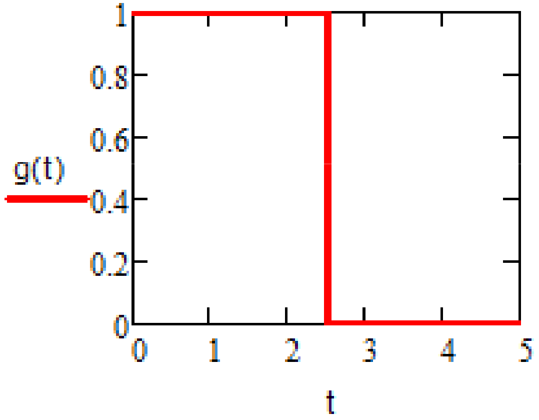
The use of the procedure described to be demonstrated by the computed in sw Mathcad

Example – response to the rectangular forcing function

Numerical solution for Fourier coefficients:

$$f(t) = h \qquad \text{pro} \qquad 0 \le t < \frac{T}{2}$$

$$f(t) = 0 \qquad \text{pro} \qquad \frac{T}{2} \le t \le T$$



$$u(t) = \frac{a_0}{2} + \sum_{k=1}^{oo} \left(a_k \cdot \cos(k \cdot \omega \cdot t) + b_k \cdot \sin(k \cdot \omega \cdot t)\right) \qquad \text{kde}$$

$$a_0 = \frac{2}{T} \cdot \int_0^{\frac{T}{2}} u(t)\, dt$$

$$a_k = \frac{2}{T} \cdot \int_0^{\frac{T}{2}} u(t) \cdot \cos(k \cdot \omega \cdot t)\, dt \qquad \text{pro } k = 1,2,\ldots$$

$$b_k = \frac{2}{T} \cdot \int_0^{\frac{T}{2}} u(t) \cdot \sin(k \cdot \omega \cdot t)\, dt \qquad \text{pro } k = 1,2,\ldots$$

Numerical solution for Fourier coefficients

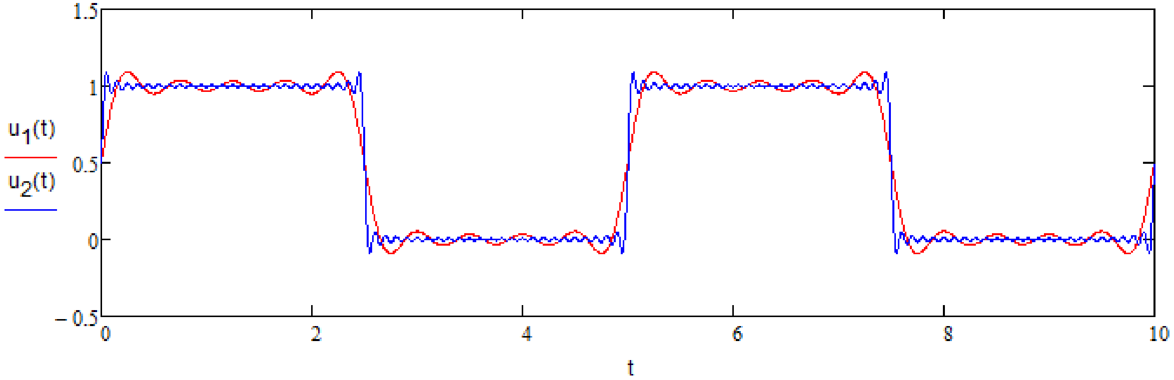
Výpočet pro k_{max}=10

$k_{max} := 10$ $T := 5$ $h := 1$ $t := 0, 0.005 .. 10$

$k := 1 .. k_{max}$ $\omega := \frac{2\cdot\pi}{T}$

$a_0 := h$ $a_k := \frac{2}{T}\cdot\frac{\sin\left(\frac{1}{2}\cdot T\cdot k\cdot\omega\right)}{k\cdot\omega}\cdot h$ $b(k) := (-h)\cdot\frac{\cos(k\cdot\pi) - 1}{k\cdot\pi}$

$$u_1(t) := \frac{a_0}{2} + \sum_{k=1}^{k_{max}} \left(a_k\cdot\cos(k\cdot\omega\cdot t) + b(k)\cdot\sin(k\cdot\omega\cdot t)\right)$$



Výpočet pro k_{max}=50

$k_{mx} := 50$

$k := 1 .. k_{mx}$

$a_k := \frac{2}{T}\cdot\frac{\sin\left(\frac{1}{2}\cdot T\cdot k\cdot\omega\right)}{k\cdot\omega}\cdot h$ $b_k := (-h)\cdot\frac{\cos(k\cdot\pi) - 1}{k\cdot\pi}$

$$u_2(t) := \frac{a_0}{2} + \sum_{k=1}^{50} \left(a_k\cdot\cos(k\cdot\omega\cdot t) + b_k\cdot\sin(k\cdot\omega\cdot t)\right)$$

Ustálená odezva funkce g(t)

Ustálená odezva

$$F_0 := \frac{a_0}{2}$$

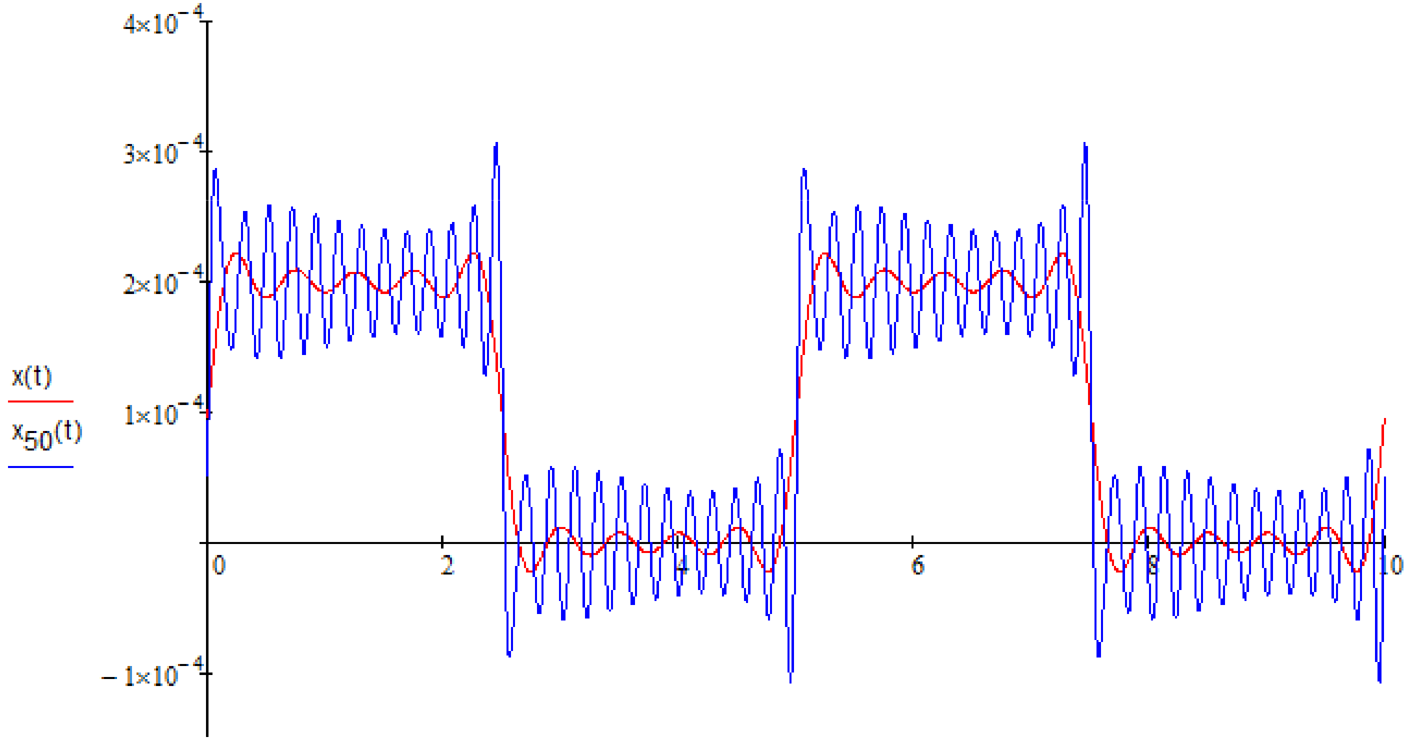
$$F_k := \sqrt{\left[\left(a_k\right)^2 + \left[\left(b_k\right)^2\right]\right]}$$

$$\psi_k := \operatorname{atan}\left[\frac{2\cdot\frac{b0}{2\cdot\sqrt{k1\cdot m}}\cdot\frac{k\cdot\omega}{\sqrt{\frac{k1}{m}}}}{1-\left(\frac{k\cdot\omega}{\sqrt{\frac{k1}{m}}}\right)^2}\right]$$

$$x(t) := \frac{F_0}{k1} + \sum_{k=1}^{10}\left[\left[\frac{b_k}{k1}\cdot\frac{1}{\sqrt{\left[1-\left(\frac{k\cdot\omega}{\sqrt{\frac{k1}{m}}}\right)^2\right]^2+\left(2\cdot\frac{b0}{2\cdot\sqrt{k1\cdot m}}\cdot\frac{k\cdot\omega}{\sqrt{\frac{k1}{m}}}\right)^2}}\right]\cdot\sin\left(k\cdot\omega\cdot t-\psi_k\right)\right]$$

$$x_{50}(t) := \frac{F_0}{k1} + \sum_{k=1}^{50}\left[\left[\frac{b_k}{k1}\cdot\frac{1}{\sqrt{\left[1-\left(\frac{k\cdot\omega}{\sqrt{\frac{k1}{m}}}\right)^2\right]^2+\left(2\cdot\frac{b0}{2\cdot\sqrt{k1\cdot m}}\cdot\frac{k\cdot\omega}{\sqrt{\frac{k1}{m}}}\right)^2}}\right]\cdot\sin\left(k\cdot\omega\cdot t-\psi_k\right)\right]$$

Ustálená odezva funkce g(t)



Průběh ustálené odezvy na funkci g(t) pro prvních k-členů řady

-------- k = 10; -------- k = 50