Vibration of Elastic Bodies

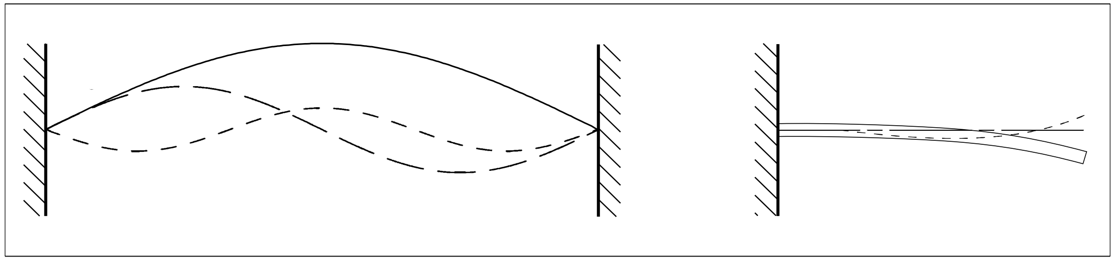
Iva Petríková

Dept. of Applied Mechanics

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

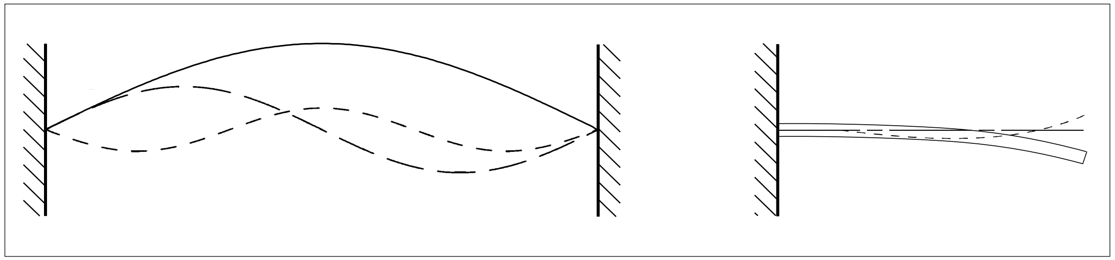
Vibration of linear continuous system

- 1 Lateral vibration of beam
- 2 Sample examples
- 3 Torsional vibration of rods



Vibration of linear continuous system

- Continuous system = homogenous physical of elastic body with continuously distributed masses and elasticity
- Single dimensional continuous system – string, rod, beam, shaft
- Linear model – applicable for small deformation
- Homogenous and isotropic materials - described by density ρ material moduli constants G and E, (viscous damping)



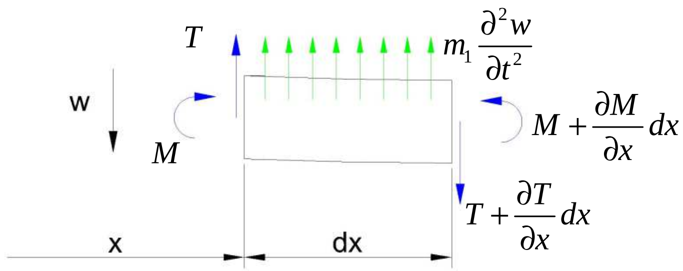
Lateral vibration of beam

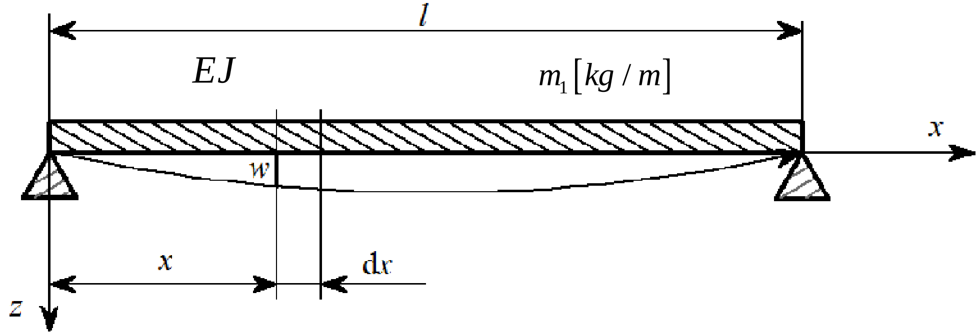
Deflection of beam as a function of position x and time t $w = w(x,t)$

Assumption: prismatic beam

Mass per unit length of beam $m_1 \left[kg/m\right]$

Bending stiffness EJ





Differential equation of the deflection curve

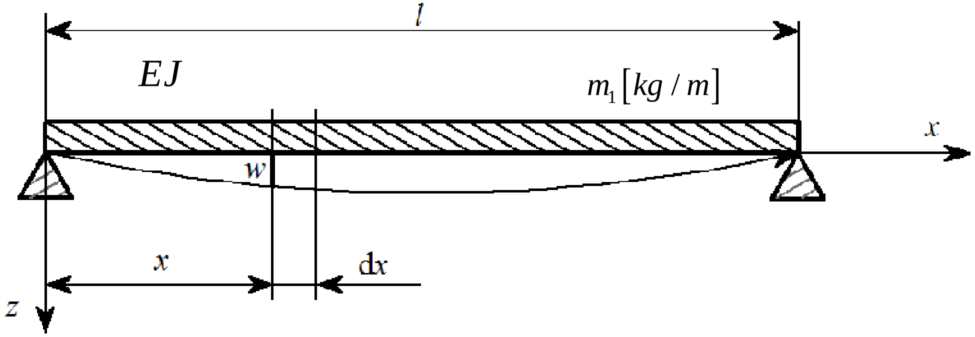
$$w^{IV}(x) = \frac{q(x)}{EJ}$$

$$\frac{\partial^4 w(x,t)}{\partial x^4} = \frac{q(x)}{EJ} \qquad (1)$$

Distributed load (inertia force of the beam in lateral direction)

$$\Rightarrow q(x,t) = -m_1 \frac{\partial^2 w(x,t)}{\partial t^2} \qquad (2)$$

Lateral vibration of beam



- Equation of motion is set up from eq. (1) and (2)
- Diferential equation (DE) of bending vibration of the beam is a partial diferential equation of the 4th order.

$$\frac{\partial^4 w(x,t)}{\partial x^4} + \frac{m_1}{EJ} \cdot \frac{\partial^2 w(x,t)}{\partial t^2} = 0$$

Fourier´s method of solution of DE

- Solution $w(x,t)=w_1(x)\cdot w_2(t)$ (3)
- $w_2(t)$ ... harmonic function

$$w(x,t)=w_1(x)\cdot(A\cos\Omega t+B\sin\Omega t)$$

- Defferentiation of eq. (3) with respect to x

$$\frac{\partial^4 w(x,t)}{\partial x^4}=\frac{\mathrm{d}^4 w_1(x)}{\mathrm{d}x^4}w_2(t)$$

- Defferentiation of eq. (3) with respect to t

$$\frac{\partial^2 w(x,t)}{\partial t^2}=-\Omega^2 w_1(x)w_2(t)$$

Fourier´s method of solution

$$\left[\frac{\mathrm{d}^4 w_1(x)}{\mathrm{d}x^4} - \Omega^2 \frac{m_1}{EJ} w_1(x)\right] w_2(t) = 0 \qquad w_2(t) \neq 0$$

$$\left[\frac{\mathrm{d}^4 w_1(x)}{\mathrm{d}x^4} - \Omega^2 \frac{m_1}{EJ} w_1(x)\right] = 0 \qquad a^4 = \Omega^2 \frac{m_1}{EJ}$$

- Differential equation of 4th order $w_1^{IV}(x) - a^4 w_1(x) = 0$
- Solution: $\lambda^4 - a^4 = 0$

$$\lambda_{1,2,3,4} = a, -a, ia, -ia$$

$w_1(x)$ is given by linear combination of all solutions

$$w_1(x) = \overline{c}_1 e^{ax} + \overline{c}_2 e^{-ax} + \overline{c}_3 e^{iax} + \overline{c}_4 e^{-iax}$$

$$w_1(x) = c_1 \cosh ax + c_2 \sinh ax + c_3 \cos ax + c_4 \sin ax$$

Boundary conditions

Deflections and bending moments for pinned ends:

$$w_1(0)=0 \qquad c_1+c_3=0$$
$$w_1''(0)=0 \qquad a^2(c_1-c_3)=0$$
$$\left.\right\} c_1=c_3=0 \quad (a\neq 0)$$

$$w_1(l)=0 \qquad c_2\sinh al+c_4\sin al=0$$
$$w_1''(l)=0 \qquad a^2(c_2\sinh al-c_4\sin al)=0$$

- Natural frequencies:

$$D=\begin{vmatrix}\sinh al & \sin al\\ \sinh al & -\sin al\end{vmatrix}=0$$

$$D=-2\sinh al\sin al=0$$
$$\sin al=0$$
$$al=\pi,2\pi,3\pi,\dots,k\pi$$
$$a_k=\frac{k\pi}{l}$$ for k = 1,2, …n

Natural frequencies

$$\Omega_k = a_k^2 \sqrt{\frac{EJ}{m_1}} = (k\pi)^2 \sqrt{\frac{EJ}{m_1 l^4}}$$

$$\Omega_1 = \pi^2 \sqrt{\frac{EJ}{m_1 l^4}}$$ 1st natural frequency

Comment: deflection w_1

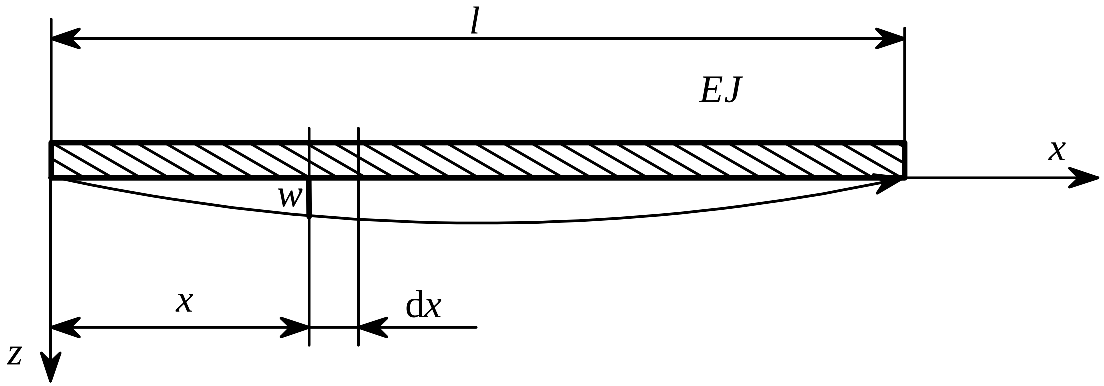
slope w_1'

bending moment $w_1'' \approx M$

shear force $w_1''' \approx T$

$$w''(x) = -\frac{M(x)}{EJ}$$

Vibration of beam (free ends)



$$\frac{\partial^4 w(x,t)}{\partial x^4} + \frac{m_1}{EJ} \cdot \frac{\partial^2 w(x,t)}{\partial t^2} = 0$$

$$w_1^{IV}(x) - a^4 w_1(x) = 0 \qquad w = w(x,t)$$

Boundary conditions

Bending moments and shear forces for free ends:

$$w_1''(0)=0 \qquad c_1a^2-c_3a^2=0 \Rightarrow c_1=c_3 \quad (a\neq 0)$$

$$w_1'''(0)=0 \qquad c_2a^3-c_4a^3=0 \Rightarrow c_2=c_4 \quad (a\neq 0)$$

$$w_1''(l)=0 \qquad c_1a^2\left(\cosh al-\cos al\right)+c_2a^2\left(\sinh al-\sin al\right)=0$$

$$w_1'''(l)=0 \qquad c_1a^3\left(\sinh al+\sin al\right)+c_2a^3\left(\cosh al-\cos al\right)=0$$

- Natural frequencies:

$$D=\begin{vmatrix}\cosh al-\cos al & \sinh al-\sin al\\ \sinh al+\sin al & \cosh al-\cos al\end{vmatrix}=0 \quad 1)$$

$$\cosh^2 al-2\cosh al\cos al+\cos^2 al-\sinh^2 al+\sin^2 al=0$$

$$D = 2 - 2\cosh al \cos al = 0 \qquad \cos al = \frac{1}{\cosh al} \quad (*)$$

Numerical solution of equation (*):

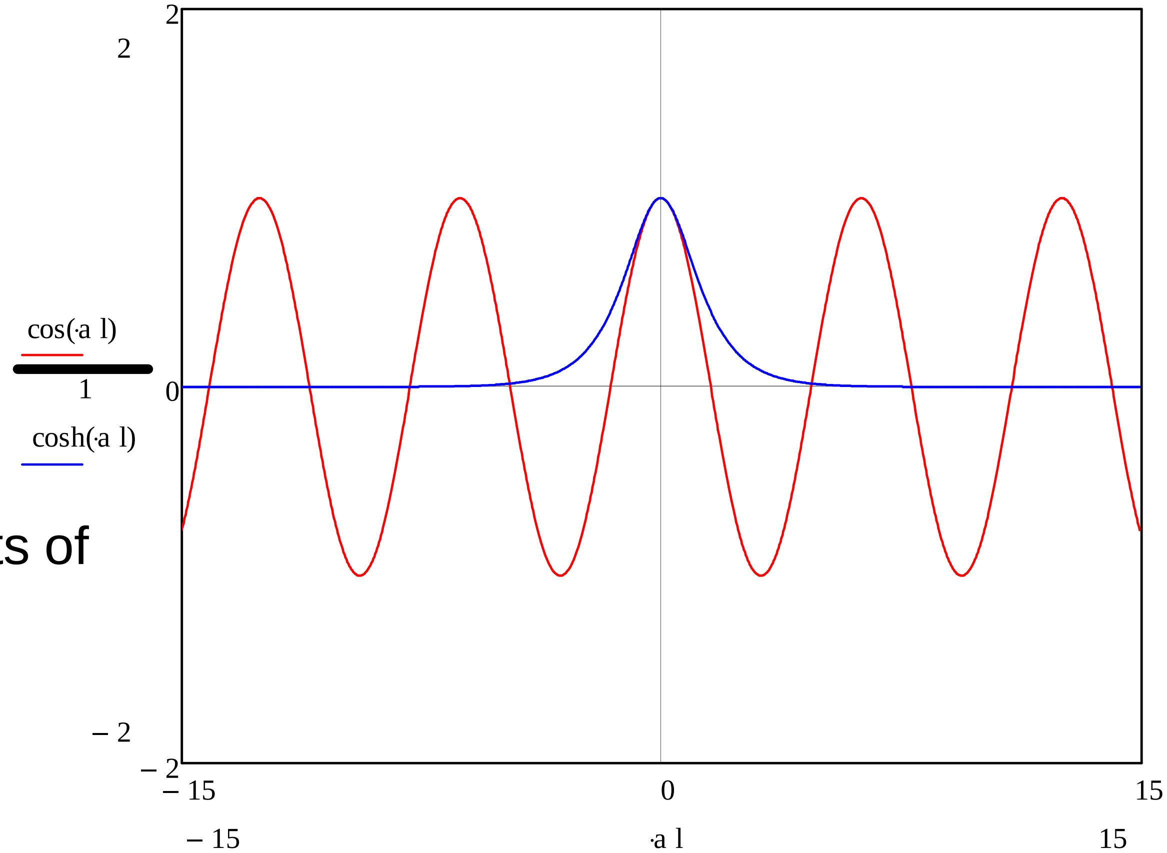
al = 0, 4.73, 7.853, 10.996, 14.137.

Natural frequencies are calculated:

$$\Omega_k = a_k^2 \sqrt{\frac{EJ}{m_1}}$$

Solutions is given by intersection points of two curves:

$$\cos al, \quad \frac{1}{\cosh al}$$



Torsional vibration of rods

Given: density ρ [kg/m^3], shear modulus G [Pa], polar moment of inertia J_p [m^4], mass moment of inertia per unit length I_1 [kg·m^2/m]

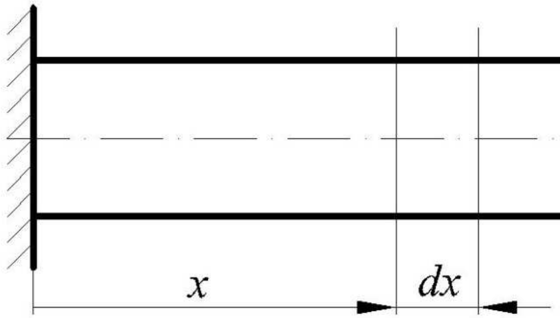


Fig. 1

$$\varphi = \varphi(x,t)$$

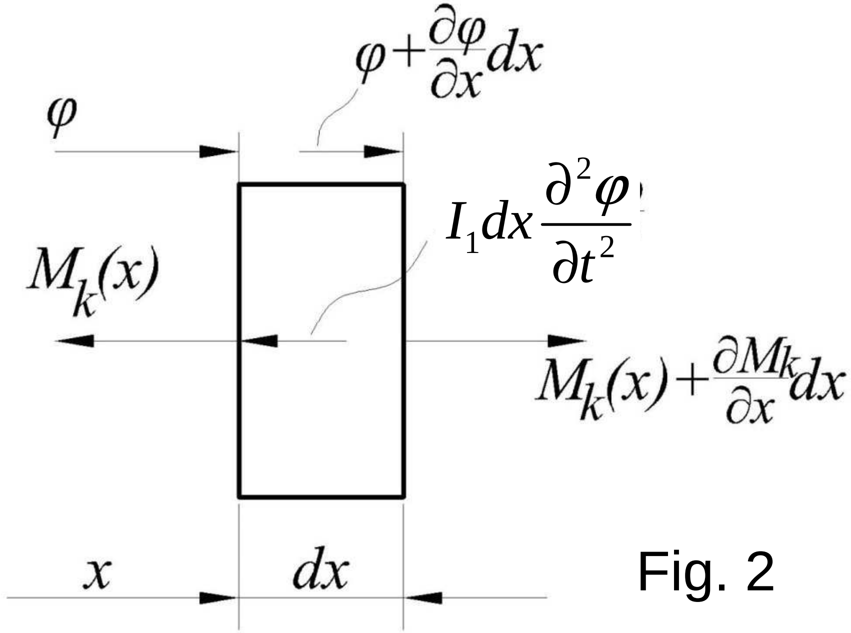


Fig. 2

According to Fig. 2 the equation of equilibrium is set up :

$$-M_k - I_1 dx \frac{\partial^2 \varphi}{\partial t^2} + M_k + \frac{\partial M_k}{\partial x} dx = 0\,,$$ M_k is torque

$$-I_1 dx \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial M_k}{\partial x} dx = 0$$

$$\frac{\partial \varphi}{\partial x} = \frac{M_k}{GJ_p} \quad \Rightarrow \quad M_k = GJ_p \frac{\partial \varphi}{\partial x} \quad \rightarrow \quad \frac{\partial M_k}{\partial x} = GJ_p \frac{\partial^2 \varphi}{\partial x^2}$$

ρ

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{I_1}{G \; J_p} \frac{\partial^2 \varphi}{\partial t^2} = 0,$$ where ρ is density

$$\frac{\partial^2 \varphi(x,t)}{\partial x^2} - \frac{\rho}{G} \frac{\partial^2 \varphi(x,t)}{\partial t^2} = 0$$

Solution of the partial differential equation:

$$\varphi(x,t) = \varphi_1(x)\cdot\varphi_2(t) = \varphi_1(x)\left(A\cos\Omega t + B\sin\Omega t\right)$$

$$\frac{\partial^2\varphi(x,t)}{\partial x^2} = \varphi_1''(x)\cdot\varphi_2(t)$$

$$\frac{\partial^2\varphi(x,t)}{\partial t^2} = \varphi_1(x)\cdot\left(-\Omega^2\right)\cdot\underbrace{\left(A\cos\Omega t + B\sin\Omega t\right)}_{\varphi_2(t)}$$

$$\varphi_1''(x)\varphi_2(t) + \frac{\rho}{G}\Omega^2\varphi_1(x)\varphi_2(t) = 0$$

$$\left[\varphi_1''(x) + \underbrace{\frac{\rho}{G}\Omega^2}_{\lambda^2}\varphi_1(x)\right]\varphi_2(t) = 0 \qquad \lambda = \Omega\sqrt{\frac{\rho}{G}} \quad\Rightarrow\quad \Omega = \lambda\sqrt{\frac{G}{\rho}}$$

Solution: $\varphi_1 = c_1 \cos \lambda x + c_2 \sin \lambda x$

Boundary conditions:

for $x = 0: \quad \varphi_1 = 0 \quad \Rightarrow \quad c_1 = 0$

for $x = l: \quad M_k(l) = 0 \quad \Rightarrow \quad M_k = GJ_p \dfrac{\partial \varphi}{\partial x} \quad \Rightarrow \quad \left. \dfrac{\partial \varphi_1(x)}{\partial x} \right|_{x=l} = 0$

$$\frac{\partial \varphi_1(l)}{\partial x} = (-c_1 \sin \lambda x + c_2 \cos \lambda x)\lambda$$

$$\varphi_1(l) = 0$$

$$\varphi_1'(x) = -c_1\lambda\sin\lambda x + c_2\lambda\cos\lambda x$$

$$\varphi_1'(x) = c_2\lambda\cos\lambda x$$

$$\varphi_1'(l) = c_2\lambda\cos\lambda l = 0, \quad c_2 \neq 0, \lambda \neq 0 \Rightarrow \cos\lambda l = 0 \qquad \lambda l = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\lambda_k = (2k-1)\frac{\pi}{2l}$$

Natural frequencies: $\Omega_k = (2k-1)\frac{\pi}{2l}\sqrt{\frac{G}{\rho}}$, pro k=1, 2, 3, …

1st natural frequency: $\Omega_1 = \frac{\pi}{2l}\sqrt{\frac{G}{\rho}}$